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# Packing of graphs

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## CONTENTS




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*To Romana*

## Preface

There are two basic reference texts on packing theory: the last chapter of Bollobás's book [6] (1978) and the 4th chapter of Yap's book [85] (1986). They still remain the main references to packing problems. However, many papers related to these problems have recently been published and the reason for writing this survey is to gather in a systematic form results scattered throughout the literature.

I wish I could name all who deserve my thanks. I am particularly grateful to A. P. Wojda for introducing me to graph theory, and to Z. Skupień for his interest in my research and for many stimulating conversations.

During my work, I had the opportunity to stay for some time at *Laboratoire de Recherche en Informatique* (*Orsay, Université Paris-Sud*), where a part of this survey was written. I would like to thank all members of *Equipe Graphes* from Orsay for the invitation and hospitality. This stay was partially supported by TEMPUS grant IMG-93-PL-1220.

Finally, I wish to express my deep gratitude to the referee for numerous helpful comments.

## 1. Introduction

The mathematical study of packing problems was initiated by a series of papers published in the late seventies. The first result about packing of a graph with itself into a complete graph was published in 1977. This theorem is important because it can be generalized in a great variety of ways. Some of these generalizations will be presented below. Since 1978 more than 70 papers have been written on this and related subjects.

This survey does not cover *all* the work in this field. It deals with those aspects of the packing theory that are of particular interest to the author and is intended as a complement of the existing texts mentioned in the Preface.

For the convenience of the reader, in this chapter we present some graph-theoretic preliminaries that are used later.

Chapter 2 begins with the basic result mentioned above. However, the main goal of this chapter is to show some interesting relationships between the embedding of graphs in their complements and the structure of the embedding permutation. Section 2 deals with self-complementary permutations. In Section 3 the permutation structure is used, in particular, to obtain another, very simple proof of the basic result, as well as an algorithm for finding an embedding. In Section 4 we define a property of an embedding which is related to the graph. In Section 5 we consider the problem of the uniqueness of the embedding.

Chapter 3 offers results on packing of two graphs. We begin with a theorem on packing of two graphs of small size, which is an immediate generalization of Theorem 2.1. In Section 2 we present some properties which hold for large $n$. Other conditions on graphs, ensuring the existence of a packing related to the product of sizes or maximal degrees, are presented in Section 3. Section 4 deals with the packing of two graphs with small sum of sizes. Section 5 is devoted to the well-known Erdős–Sós Conjecture. Some other problems related to trees and forests are treated in Section 6. The final section of that chapter contains some generalizations.

In Chapter 4 we discuss the packing of three graphs. It turns out that under the same assumptions as in the basic Theorem 2.1 we can pack (with few exceptions) not only two but three copies of a graph into a complete graph. Section 1 provides the first proof of this fact. The second proof is given in the next section, where we establish the cycle structure of a packing permutation. A similar problem related to a tree is treated in Section 3. In Section 4 we investigate the problem of packing of three trees. In Sections 5 and 6 we give two general theorems on the packing of three trees and forests, respectively.

Chapter 5 deals with some problems of a special nature. It begins with one more improvement of the basic theorem (Section 1). In Section 2 we discuss the packing of two copies of a graph into the graph $K_n \setminus C_n$. We conclude with a brief description of other packing problems. In Section 3 we consider sequences of trees. In Section 4 we summarize without proofs the results concerning the packing into bipartite graphs. In the last section we review some facts on packing of digraphs.

**1.1. Basic graph-theoretic terms.** Unfortunately, the notation in graph literature is far from being unified. The reader who is unfamiliar with the standard graph theory concepts may consult [2], [7], or [9], for example.

A *graph* $G = (V, E)$ consists of a finite nonempty set $V = V(G)$ of *vertices*, and a family $E = E(G)$ of two-element subsets of $V$ called *edges*.

Given a graph $G = (V, E)$, the number of vertices in $V$ is called the *order* of $G$ and the number of edges in $E$ is called the *size* of $G$. They will be denoted by $|V|$ and $|E|$, respectively. We also write $e(G)$ for the size of $G$. If a graph $G$ has order $n$ and size $m$, we say that $G$ is an $(n, m)$ graph. We shall also use the notation $G = G(n, m)$.

Two vertices that are joined by an edge are called *adjacent*, as are two edges that meet at a vertex. If two vertices are not joined by an edge, we say that they are *nonadjacent* or *independent*. Similarly, two edges that not share a common vertex are said to be *independent*.

If a vertex $u$ is connected by an edge with a vertex $v$, we say that $u$ is a *neighbour* of $v$. The set of all vertices adjacent to a vertex $v$ is called the *neighbourhood* of $v$ and is denoted by $N_G(v)$, or simply by $N(v)$ if there is no danger of confusion.

The edge between the vertices $u$ and $v$ (denoted by $uv$) is said to be *incident* with $u$ (or with $v$). The number of edges incident with a vertex $v$ is called the *degree* (or *valency*) of $v$ and is denoted by $d(v)$ or by $d_G(v)$ if we wish to emphasize the dependence on the graph $G$. The minimum and the maximum degrees of a vertex in the graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Two graphs $G_1$ and $G_2$ are *isomorphic* if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$ (that is, $f$ preserves adjacency and nonadjacency).

A graph $H$ is a *subgraph* of a graph $G$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A subgraph $H$ of $G$ such that whenever $u, v \in V(H)$ are adjacent in $G$ then they are also adjacent in $H$, is called an *induced subgraph* of $G$.

Let $H$ be a subgraph of a graph $G$. The set of all vertices of $H$ adjacent to a vertex $v$ is denoted by $N_G(v, H)$, or simply by $N(v, H)$ if there is no danger of confusion. The set of edges of $G$ incident with at least one vertex of $H$ is said to be *covered* by $H$.

A vertex of valency 0 is called an *isolated vertex*. A vertex of valency 1 is called a *pendent vertex* or an *end-vertex*. An edge incident with such a vertex is called an *end-edge* or a *pendent edge*. The maximum valency of a vertex in a graph of order $n$ is clearly $n-1$. A graph of order $n$ having all vertices of maximum valency is called a *complete graph* and is denoted by $K_n$.

A graph of order $p + q$ with two disjoint sets of independent vertices $X$ and $Y$, $|X| = p$, $|Y| = q$ (called *colour classes*) that contains all edges between $X$ and $Y$ is called a *complete bipartite graph* and is denoted by $K_{p,q}$. A *complete r-partite graph* is defined analogously.

Let $x$ and $y$ be two vertices of a graph $G$. A *path* joining $x$ and $y$ is an alternating sequence of vertices and edges $x_0, e_1, x_1, e_2, \ldots, e_k, x_k$ of the graph $G$ in which no vertex is repeated and such that $x_0 = x$, $x_k = y$ and $e_i = x_{i-1}x_i \in E(G)$ for $i = 1, 2, \ldots, k$. We then say that the path is of *length* $k$. We denote the path by $x_0x_1 \ldots x_k$. If $k \geq 2$ and $xy \in E(G)$, then a path joining $x$ and $y$ is said to be *closed*. A closed path is also called a *cycle*. A graph of order $n$ that consists only of a cycle (or a path) is denoted by $C_n$ (or $P_n$).

We say that a graph $G$ is *connected* if there exists a path in $G$ between any two of its vertices and we say that $G$ is *disconnected* otherwise. A *component* of a graph is a maximal connected subgraph.

The *distance* $\text{dist}(u, v)$ between two vertices $u$ and $v$ of $G$ is the length of a shortest path joining $u$ and $v$. The *diameter* $\text{diam}(G)$ is the maximum distance $\text{dist}(u, v)$ taken over all pairs $(u, v)$ of vertices of $G$. The length of a shortest cycle in $G$ is called the *girth* of $G$ and the length of a longest cycle in $G$ is called the *circumference* of $G$. A cycle that includes all vertices of $G$ is called a *hamiltonian cycle* of $G$. If $G$ has a hamiltonian cycle, then $G$ is said to be *hamiltonian*.

There are many ways to construct new graphs.

The *complement* $\overline{G}$ of a graph $G$ is the graph with the vertex set $V(G)$ and such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in $G$.

Assume now that $G_1$ and $G_2$ are two graphs with disjoint vertex sets.

The *union* $G = G_1 \cup G_2$ is a graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of $n$ $(\geq 2)$ disjoint copies of a graph $H$, then we write $G = nH$.

The *join* $G = G_1 + G_2$ (or $G = G_1 * G_2$) consists of $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

For our next operation, the conditions are quite different. Let now $G_1$ and $G_2$ be graphs with $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. The *edge sum* $G_1 \oplus G_2$ has $V(G) = V(G_1) = V(G_2)$ as the set of vertices and $E(G) = E(G_1) \cup E(G_2)$ as the set of edges.

A graph obtained from $G$ by deleting one or more of its vertices, that is, the subgraph induced by $V(G) \setminus \{v_1, \ldots, v_k\}$, is denoted by $G \setminus \{v_1, \ldots, v_k\}$ or by $G - v_1 - \ldots - v_k$. By $G - e$ we denote the graph obtained from $G$ by deleting one edge and by $G + e$ we denote the graph obtained from $G$ by adding one edge.

**1.2. Some families of graphs.** If a graph contains no cycles it is called *acyclic* or a *forest*. We define a *tree* to be a connected acyclic graph; so a forest is a graph whose each component is a tree. A vertex of degree at most one in a forest is called a *leaf*. It is worthwhile to remark that each nontrivial tree has at least two leaves. Some families of trees will often appear in our considerations.

A *star* $S_n$ is a tree that contains one vertex of degree $n-1$ (the centre) and $n-1$ vertices of degree 1. In other words, $S_n$ is the complete bipartite graph $K_{1,n-1}$.

$S'_n$ is a tree obtained from $S_{n-1}$ by inserting a new vertex on an edge and $S''_n$ is a tree obtained from $S_{n-2}$ by inserting two new vertices on one edge.

A *double star* is a graph obtained from two stars by joining their centres by an edge. More generally, a tree obtained from two stars by joining their centres by a path is called a *star-path-star* (an *s-p-s*).

A *comet* is a tree obtained from a star and a path by identifying one leave of the star with one leave of the path.

A vertex of a tree having valency at least two is an *interior* vertex. A tree is a *caterpillar* if the induced subgraph of its interior vertices is a path.

We denote by $T_r(n)$ the Turán graph, that is, the complete $r$-partite graph with colour classes as close as possible, i.e. with $\lfloor n/r \rfloor, \lfloor (n+1)/r \rfloor, \ldots, \lfloor (n+r-1)/r \rfloor$ vertices in the colour classes. We put $t_r(n) = e(T_r(n))$. Clearly, $T_r(n)$ does not contain the complete graph $K_{r+1}$ and the *theorem of Turán* states that every graph with $n$ vertices and more than $t_r(n)$ edges contains a copy of $K_{r+1}$.

A graph $G$ is *self-complementary* (briefly, *s-c*) if it is isomorphic to its complement (cf. [51], [52], or [36]). It is clear that an s-c graph has $n \equiv 0, 1 \pmod 4$ edges. We extend the above definition to the case where $n \equiv 2, 3 \pmod 4$ as follows. A graph $G$ of order $n \equiv 2, 3 \pmod 4$ is *almost self-complementary* (or briefly, *a-s-c*) if $G$ is of size $\frac{1}{2}\left(\binom{n}{2} - 1\right)$ and $G$ is a subgraph of its complement.

By analogy to the definitions of s-c and a-s-c graphs, a graph $G$ of order $n \equiv 0, 1 \pmod 3$ is a 3-*s-c graph* if $G$ is of size $\frac{1}{3}\binom{n}{2}$ and the complete graph $K_n$ can be decomposed into three edge-disjoint graphs each of them isomorphic to $G$.

A graph $G$ of order $n \equiv 2 \pmod 3$ is a 3-*a-s-c graph* if $G$ is of size $\frac{1}{3}\left(\binom{n}{2} - 1\right)$ and the graph $K_n - e$ can be decomposed into three edge-disjoint graphs each of them isomorphic to $G$ (see also [62], [40] for some related problems).

**1.3. Edge-disjoint placements of graphs.** Suppose $G_1, \ldots, G_k$ are graphs of order less than or equal to $n$. We say that there is a *packing* of $G_1, \ldots, G_k$ (into the complete graph $K_n$) if there exist injections $\alpha_i : V(G) \to V(K_n),\ i = 1, \ldots, k$, such that

$$\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G)) \neq \emptyset \quad \text{for } i \neq j,$$

where the map $\alpha_i^* : E(G_i) \to E(K_n)$ is induced by $\alpha_i$.

A packing of $k$ copies of a graph $G$ will be called a *$k$-placement* of $G$. A packing of two copies of $G$, i.e. a 2-placement, is an *embedding* of $G$ (in its complement $\overline{G}$). So, an embedding of a graph $G$ is a permutation $\sigma$ on $V(G)$ such that if an edge $xy$ belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

It is easy to see that there exists a packing of two graphs $G_1$ and $G_2$ if and only if $G_1$ is a subgraph of $\overline{G}_2$ (or, by symmetry, $G_2$ is a subgraph of $\overline{G}_1$). However, we wish to distinguish between the problems about packings and the problems about the existence of certain subgraphs. In "packing" problems we claim that each member of a *large* family of graphs contains each member of another *large* family. In "subgraph" problems usually at least one of the two graphs is fixed.

## 2. Embeddings of graphs

**2.1. Basic result.** The following theorem was proved, independently, in [8], [16], and [57].

THEOREM 2.1. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n - 2$ then $G$ can be embedded in its complement $\overline{G}$.*

Proof. We employ induction on $n$. The theorem is easily seen to be true for $n = 2, 3$ or 4 by inspecting all graphs in question. Suppose that the theorem is true for all $n' < n$, $n \geq 5$. Consider a graph $G$ of order $n$ and size $e(G) = n - 2$. We distinguish two cases.

Case 1: $G$ contains an isolated vertex $x$. Then $G$ must contain a vertex $y$ of degree $d_G(y) \geq 2$. Thus the induction assumption is directly applicable to the graph $G'$ defined by $G' = G - \{x, y\}$. Hence there exists an embedding $\sigma'$ of $G'$ into its complement.

Now, putting $\sigma(x) = y$, $\sigma(y) = x$ and $\sigma(v) = \sigma'(v)$ for $v \in V(G')$ we obtain an embedding of $G$ into $\overline{G}$.

Case 2: $G$ contains no isolated vertices. Observe that $G$ must contain at least two components that are trees. Moreover, these trees are nontrivial. Hence it is easy to choose two independent end-edges in $G$, say $xx'$ and $yy'$, by choosing, for instance, the corresponding end-vertices $x$ and $y$ in different trees of $G$. Consider the graph $G' = G \setminus \{x, y\}$. By the induction hypothesis there exists an embedding $\sigma'$ of $G'$.

It now suffices to observe that we can always extend $\sigma'$ and obtain an embedding $\sigma$ of $G$. For example, in the "worst" case, where $\sigma'(x') = x'$ and $\sigma'(y') = y'$, we put $\sigma(x) = y$ and $\sigma(y) = x$ (see Fig. 2.1).

Thus, by induction, the proof is complete. ■

Remark. The star $K_{1,n-1}$ provides an example of an $(n, n-1)$ graph that is not contained in its complement. Thus the above theorem is the "best possible" in the sense

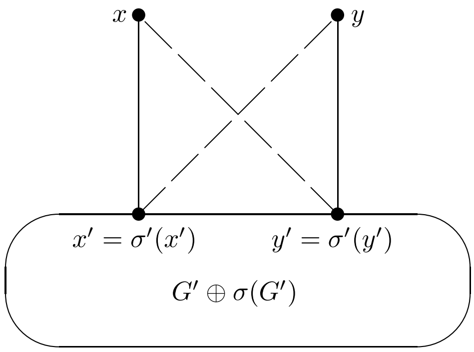



Fig. 2.1. The "worst" case

that it cannot be improved by increasing the size of the graph $G$.

H. J. Straight [65] has observed that the star is the only tree not contained in its complement. Some improvements of this result will be given later (cf. for instance Theorem 2.8).

Theorem 2.1 has been improved in many ways. In the next sections of this chapter we present, in particular, those improvements that in addition to the existence provide more information about embeddings.

**2.2. elf-complementary permutations.** The following theorem, originally proved in [17], completely characterizes those graphs with $n$ vertices and $n-1$ edges that are embeddable.

THEOREM 2.2. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n - 1$ then either $G$ is embeddable or $G$ is isomorphic to one of the following graphs*: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ *with* $n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$ (see Fig. 2.2). ■

R e m a r k. Note that the graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$ are embeddable.

The proof we present below uses the cyclic structure of the packing permutation. Actually, we shall improve Theorem 2.2 by showing that there exists a packing which is an s-c permutation or an a-s-c permutation. The part concerning the cases $n \equiv 0, 1 \pmod 4$ was proved in [4]. The existence of an almost-self-complementary packing (i.e. the cases $n \equiv 2, 3 \pmod 4$) was proved in [79].

More precisely, we shall prove the following theorem.

THEOREM 2.3. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n - 1$ then either there exists an embedding $\sigma$ of $G$ having all its cycles of length* 4, *except for*

- *one cycle of length one if* $n \equiv 1 \pmod 4$,
- *one cycle of length two or two cycles of length one if* $n \equiv 2 \pmod 4$,
- *two cycles of lengths one and two if* $n \equiv 3 \pmod 4$,

*or $G$ is isomorphic to one of the following graphs*: $K_{1,n-1}$, $K_{1,n-4} \cup K_3, n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.

P r o o f. Let $G$ be a graph of order $n$. Throughout this proof we shall use the following

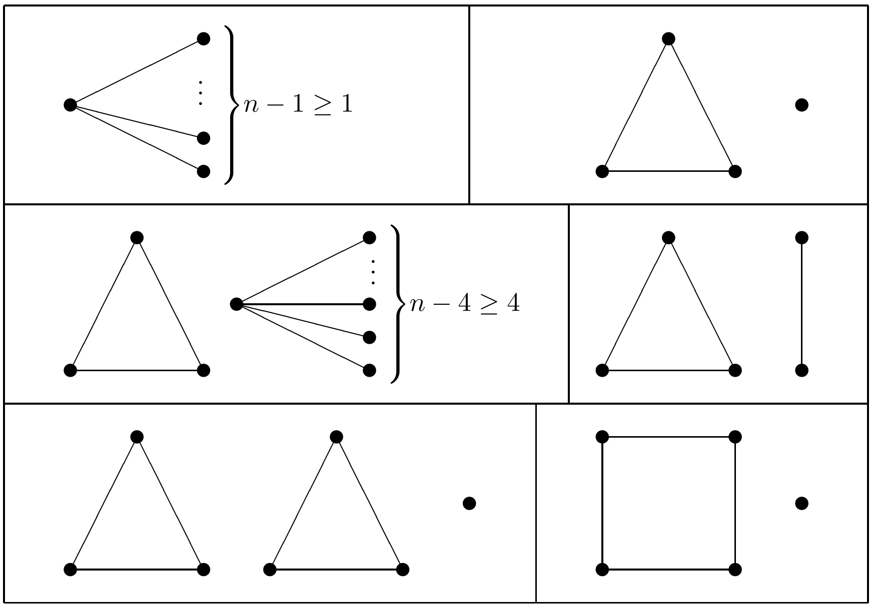



Fig. 2.2. Non-embeddable $(n, n-1)$ graphs

terminology. A permutation $\sigma$ of $V(G)$ will be called *good* for $G$ if $\sigma$ is an embedding of $G$ and all its cycles have length 4, except for one cycle of length one if $n \equiv 1 \pmod 4$, one cycle of length two or two cycles of length one if $n \equiv 2 \pmod 4$, two cycles of lengths one and two if $n \equiv 3 \pmod 4$.

Using this terminology, the theorem says that if $|E(G)| \leq n-1$, then either there exists a good permutation for $G$ or $G$ is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ with $n \geq 8$, $K_1 \cup C_4$, $K_1 \cup K_3$, $K_1 \cup 2K_3$ or $K_2 \cup K_3$.

The proof is by induction on $n$. Without loss of generality we can assume that $|E(G)| = n-1$ and that $G$ is not one of the exceptional graphs. The theorem is true for $n = 2, 3, 4$, so let $n \geq 5$, and suppose that the assertion is true for all $n' < n$. Note that $|E(G)| < n$ implies that $G$ has at least one connected component which is a tree.

We shall consider four main cases.

Case 1: $n \equiv 1 \pmod 4$

Subcase 1(a): $G$ has at least one end-vertex $x$. Consider the graph $G' = G \backslash \{x\}$. Suppose that $G'$ is not one of the exceptional graphs. Then there exists a good permutation $\sigma'$ for $G'$. Putting $\sigma(x) = x$ and $\sigma(v) = \sigma'(v)$ for $v \in V(G')$ we get a good permutation for $G$. Suppose now that $G'$ is one of the exceptional graphs. Moreover, we can assume that it is not possible to choose an end-vertex in such a way that $G'$ is not exceptional. There is only one graph satisfying this condition. Its packing is given in Figure 2.3.

Subcase 1(b): Suppose now that (a) does not hold, which implies that $G$ has at least one isolated vertex $x$. If $G$ has another isolated vertex $y$, then it is easy to

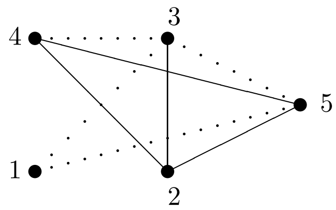



Fig. 2.3. Embedding permutation is given by (1234)(5)

find two vertices $u, v$ of $G$ that cover at least four edges of $G$. Then if the graph $G' = G \backslash \{x, y, u, v\}$ is not exceptional, we can complete a good permutation $\sigma'$ for $G'$, by putting $\sigma = \sigma'(xuyv)$.

So, we can assume that $x$ is the unique isolated vertex of $G$. This implies, in particular, that the other vertices of the graph are of valency two. In other words, $G$ is the union of an isolated vertex and some vertex-disjoint cycles.

Denote by $a_1, \ldots, a_k$ the vertices lying on the longest cycle $C_k$ of $G$. If $k \geq 5$, then the graph $G' = G \setminus \{a_1, a_2, a_3, a_4\}$ is not exceptional. So we can apply the induction hypothesis to the graph $G'$ and next obtain the required embedding for $G$ by adding the cycle $(a_1a_2a_4a_3)$ (the permutation cycle!) to the good permutation for $G'$.

If $k = 3$ or 4 then observe that since $n \equiv 1 \pmod 4$, $G$ contains either two cycles of length 4 or four cycles of length 3. We proceed as follows:

If $G$ contains two cycles of length 4, then we can pack them as indicated in Figure 2.4 and use the induction hypothesis for the remaining part of the graph $G$. Note that this is always possible except for $n = 13$ and $G = K_1 \cup\ 3C_4$. This case is left to the reader.

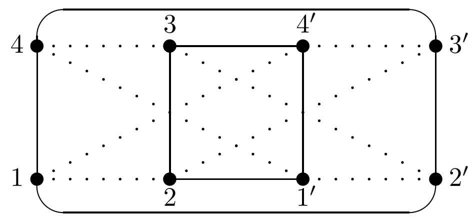



Fig. 2.4. Embedding permutation: $(1234)(1'2'3'4')$

We proceed similarly in the case where $G$ has four cycles of length 3. An embedding of $4K_3$ is indicated in Figure 2.5.

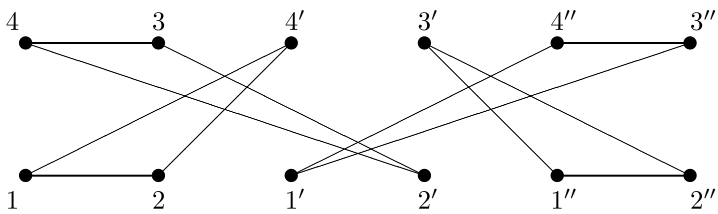



Fig. 2.5. Embedding permutation: $(1234)(1'2'3'4')(1''2''3''4'')$

Case 2: $n \equiv 2 \pmod 4$

Subcase 2(a): $G$ has two nonadjacent end-vertices $x$, $y$. Consider the graph $G' = G\backslash\{x, y\}$. Suppose that $G'$ is not one of the exceptional graphs. Then there exists a good permutation $\sigma'$ for $G'$. Putting $\sigma(x) = x$, $\sigma(y) = y$ and $\sigma(v) = \sigma'(v)$ for $v \in V(G')$ we get a good permutation for $G$. The case where $G'$ is one of the exceptional graphs is left to the reader.

Subcase 2(b): $G$ has an isolated vertex. Denote the isolated vertex of $G$ by $x$ and let $y$ be a vertex of $G$ with $d(y) \geq 2$. Consider the graph $G' = G\backslash\{x, y\}$ and suppose that $G'$ is not one of the exceptional graphs. Denote by $\sigma'$ a good permutation for $G'$. Then a good permutation for $G$ can be defined by $\sigma = \sigma'(xy)$. As in (a), the case where $G'$ is one of the exceptional graphs is left to the reader.

Subcase 2(c): Neither (a) nor (b) hold. Then $G$ contains an isolated edge $xy$, and all other vertices of $G$ have degree 2. Denote by $u, v$ two nonadjacent vertices of $G$ and consider the graph $G' = G\backslash\{x, y, u, v\}$. Since $G'$ is of order $n - 4$ and has $n - 6$ edges, it follows by induction that there exists a good permutation $\sigma'$ for $G'$. Define $\sigma$ by $\sigma = \sigma'(xuyv)$. It is easy to see that $\sigma$ is a good permutation for $G$.

Case 3: $n \equiv 3 \pmod 4$. If $G$ has an isolated vertex or an isolated edge we proceed similarly as in Case 1(b) or 4(a) making use of our induction hypothesis. Otherwise $G$ has a component $T$ which is a tree, $T \neq K_1,\ T \neq K_2$.

We consider two subcases.

Subcase 3(a): $T$ has three end edges $xx'$, $yy'$, $zz'$ with end-vertices $x$, $y$, $z$. We put $G' = G\backslash\{x, y, z\}$. By induction hypothesis there exists a good permutation $\sigma'$ for $G'$ having all cycles of length 4. Consider, for example, the "worst" case, where $x'$, $y'$, $z'$ are distinct and belong to the same cycle of $\sigma'$. Let $\sigma'(x') = y'$, $\sigma'(y') = z'$. Then a good permutation for $G$ can be defined by $\sigma = \sigma'(xz)(y)$.

Subcase 3(b): $T$ is a path $a_1a_2 \ldots a_k$. If $k \geq 5$, we put $G' = G\backslash\{a_1, a_2, a_3, a_4\}$ and $\sigma = \sigma'(a_1a_3a_4a_2)$.

If $k = 4$, we put $G' = G\backslash\{a_1, a_3, a_4\}$ and $\sigma = \sigma'(a_1a_3)(a_4)$.

Finally, if $k = 3$ we put $G' = G\backslash\{a_1, a_2, x\}$ and $\sigma = \sigma'(a_1x)(a_2)$, where $x$ is a vertex of $G$ such that $d(x) \geq 1$, $x \neq a_i$.

In all the above cases, $\sigma'$ is a good permutation for $G'$ whose existence follows from the induction hypothesis. It is easy to see that $\sigma$ defines a good permutation for $G$.

Case 4: $n \equiv 0 \pmod 4$

Subcase 4(a): $G$ has an isolated edge or two isolated vertices. Suppose first that $G$ has an isolated edge $xy$. Then it is easy to find two nonadjacent vertices $u, v$, with $d(u) + d(v) \geq 3$. The last condition enables us to use the induction hypothesis to the graph $G' = G \setminus \{x, y, u, v\}$ and get an embedding $\sigma'$ of $G'$ with all its cycles of length four (if $G'$ is not one of the exceptional graphs). It is easy to see that then we can extend the embedding $\sigma'$ by putting $\sigma = \sigma'(xuyv)$ and get in this way a good permutation for $G$. A similar reasoning can be applied if $G$ has two isolated vertices. The cases where $G'$ is one of the exceptional graphs are left to the reader.

S u b c a s e 4(b): $G$ has an end-vertex $x$ adjacent to a vertex $y$ of valency two. Denote by $z$ the other vertex adjacent to the vertex $y$, and by $u$ a vertex of $G$ with $d(G) \geq 2$. It is easy to see that the induction hypothesis can be applied to the graph $G' = G \setminus \{x, y, z, u\}$ and that the corresponding embedding permutation extended by the cycle $(uyzx)$ is a good permutation for $G$.

S u b c a s e 4(c): $G$ contains two independent end-edges. Denote by $xx'$ and $yy'$ two independent end-edges of $G$ with $x$, $y$ its end-vertices. Since by 4(a) we can assume that the edges $xx'$ and $yy'$ are not isolated edges and by 4(b) we can assume that $d(x') \geq 3$ and $d(x') \geq 3$, we see that four vertices $x, x', y, y'$ cover at least five edges. So we apply the induction hypothesis to the graph $G' = G \setminus \{x, x', y, y'\}$ and extend the good permutation for $G'$ by adding the (permutation) cycle $(xx'yy')$.

S u b c a s e 4(d): $G$ has two adjacent vertices of valency two with nonadjacent neighbours. Denote these two adjacent vertices by $x, y$, and let $x_1$ and $y_1$ be the other neighbours of $x$ and $y$, respectively. Note that by 4(b) we can assume that $d(x_1) \geq 2$ and $d(y_1) \geq 2$. Since $x_1y_1 \notin E(G)$, the vertices $x$, $x_1$, $y$, $y_1$ cover at least five edges and we can proceed as in the previous cases.

S u b c a s e 4(e): $G$ has one isolated vertex $u$ and one end-vertex $x$. Denote by $y$ the vertex of $G$ adjacent to $x$ and by $v$ a vertex with $d(v) \geq 2,\ v \neq y$. If $d(y) = 2$, then we apply 4(b). If $d(y) \geq 3$, then the vertices $x, y, u, v$ cover at least five edges and we can proceed as in previous cases.

S u b c a s e 4(f): $G$ is the sum of a star $K_{1,p}$ with $p \geq 3$ and some edge-disjoint cycles. Denote by $k$ the length of a longest cycle $C_k$ of $G$. If $k \geq 5$ we can apply 4(d) (cf. also Case 1(b)). So let $k \leq 4$. We shall consider here only one particular case.

Let $G = K_1 \cup C_3 \cup C_4$. Denote by $y_1$, $y_2$, $y_3$ the vertices of the triangle $C_3$, by $x_1$, $x_2$, $x_3$, $x_4$ the vertices of the quadrilateral $C_4$, and by $z$ the isolated vertex $G$. It is easy to see that the permutation defined by $\sigma = (x_1x_2y_1, y_2)(x_3x_4y_3z)$ has the required properties.

Note that the case where $G = K_2 \cup 2C_3$ has been considered in Case 4(a).

These two last examples together with the examples of the Case 1(b) enable us to easily find a good permutation in the remaining cases. The details are left to the reader.

This completes the proof of Theorem 2.3. ∎

The above theorem implies the following corollary. The proof of the first part was given in [4].

COROLLARY 2.4. *Let $G = (V, E)$ be a graph of order $n$ with $|E(G)| \leq n - 1$ such that $G$ is not isomorphic to an exceptional graph of Theorem* 2.2. *Then $G$ is contained in an s-c graph* (*for* $n \equiv 0, 1 \pmod 4$) *or $G$ is contained in an a-s-c graph* (*for* $n \equiv 2, 3$ $\pmod 4$)*.*

P r o o f. We only present the proof in the case $n \equiv 0 \pmod 4$. The proof of other cases is analogous. Hence suppose that $n \equiv 0 \pmod 4$. By Theorem 2.3 there exists a permutation of $V(G)$ with all cycles of length four that is an embedding of $G$. We shall construct two graphs $H$ and $H'$ with $V(H) = V(H') = V(G)$ using the following algorithm:

(i) For every edge $e \in E(G)$, assign the edges $e$ and $\sigma^2(e)$ to $E(H)$ and assign the edges $\sigma(e)$ and $\sigma^3(e)$ to $E(H')$.

(ii) If after step (i) there exist two vertices $x, y$ of $G$ such that the edge $f = xy$ does not belong to $E(H) \cup E(H')$, then assign it as well as the edge $\sigma^2(f)$ to $E(H)$ and the edges $\sigma(f)$ and $\sigma^3(f)$ to $E(H')$.

(iii) Repeat step (ii) until all edges of the complete graph constructed on $V(G)$ lie in $E(H) \cup E(H')$.

Clearly, the graph $H$ defined by the above method is an s-c graph and the graph $H'$ is its complement, i.e. $H' = \overline{H}$. Moreover, $\sigma$ is an s-c permutation which maps $H$ onto $H'$. Finally, we clearly have $G \subset H$. ■

**2.3. Embeddings without fixed points.** The following theorem, proved in [59], has been used in the study of embeddings of $(n, n-1)$ graphs.

THEOREM 2.5. *Let $G=(V,E)$ be a graph of order $n$. If $|E(G)| \le n-2$ then there exists an embedding $\sigma$ of $G$ in its complement such that $\sigma$ has no fixed points*, *i.e. $\sigma(x) \neq x$ for $x \in V(G)$.* ■

The above theorem cannot be improved by increasing the number of edges. Consider, for instance, the graph $K_{1,2} \cup K_3$ with the vertices as in Figure 2.6.

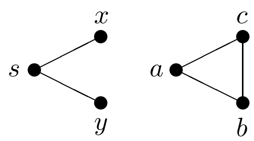


Fig. 2.6. $K_{1,2} \cup K_3$

In order to pack the triangle of the second copy of $G$ we must use the vertices $x$ and $y$ and one of the vertices of the triangle. It is now easy to see that the centre of the star of the second copy has to be mapped onto the vertex $s$, which becomes a fixed point of the embedding.

However, Theorem 2.5 can be improved in other directions. We shall present two theorems which improve it by specifying the structure of the packing permutation (cf. [79]).

THEOREM 2.6. *Let $G = (V,E)$ be a graph of order $n$ and size less than or equal to $n-2$, $n \ge 3$. If $n \neq 5$, then there exists an embedding $\sigma$ of $G$ such that each cycle of $\sigma$ has the length* 3 *or* 4. *If $n = 5$, then there exists an embedding $\sigma$ which is a cyclic permutation.*

P r o o f. It is easy to verify that the theorem holds for $n = 3, 5$. By Theorem 2.3 it also holds for $n = 4, 8$. So, we may assume that $n \ge 6$. Let $n$ be the smallest integer such that theorem does not hold. Without loss of generality we may assume that $G$ is of size $n-2$.

We shall consider several cases and subcases.

C a s e 1: $G$ has two isolated vertices $x$, $y$. If $G$ has a vertex $z$ with $d(z) \ge 3$ then, by induction, there exists an embedding $\sigma'$ of $G' = G \backslash \{x, y, z\}$ having all its cycles of length

3 or 4. Then $\sigma = \sigma'(xyz)$ is an embedding of $G$ with the same property—a contradiction. If such a vertex $z$ does not exist, then all vertices of $G$ (except $x$ and $y$) have degree 2. Let $u, v$ be two nonadjacent vertices with $d(u) = d(v) = 2$. We proceed as above with $G' = G\backslash\{x, y, u, v\}$ and $\sigma = \sigma'(xuyv)$.

Case 2: $G$ has an isolated edge $xy$. Let $z$ be a vertex of $G$ with $d(z) \geq 2$ and consider the graph $G' = G\backslash\{x, y, z\}$. Then $\sigma = \sigma'(xyz)$ is an embedding of $G$ with all cycles of length 3 or 4, where $\sigma'$ is a permutation having the same properties with respect to $G'$.

Case 3: $G$ has two nontrivial tree components $T$, $T'$. By Case 2 we can assume that neither $T$ nor $T'$ is a path of length 1. Denote by $ab$ an end edge of $T$ and by $a'b'$ an end edge of $T'$ with $a$ and $a'$ as end-vertices. Since $d(b) \geq 2$ and $d(b') \geq 2$, we can apply the induction hypothesis to the graph $G' = G\backslash\{a, b, a', b'\}$. The embedding $\sigma$ of $G$ is then defined by $\sigma = \sigma'(aa'bb')$.

Case 4: $G$ has one isolated vertex $x$ and one nontrivial tree $T$ as a component. Denote by $P = a_1a_2 \ldots a_k$ a path with the largest length in $T$. By Case 2, $k \geq 3$.

If $d(a_2) \geq 3$, then we put $G' = G\backslash\{x, a_1, a_2\}$ and $\sigma = \sigma'(xa_1a_2)$.

If $d(a_2) = 2$ and $d(a_3) \geq 3$, then we put $G' = G\backslash\{x, a_1, a_2, a_3\}$ and $\sigma = \sigma'(xa_2a_1a_3)$.

If $d(a_2) = d(a_3) = 2$ and $d(a_4) \geq 2$, then we put $G' = G\backslash\{a_1, a_2, a_3, a_4\}$ and $\sigma = \sigma'(a_1a_3a_4a_2)$.

If $T = a_1a_2a_3a_4$, then we put $G' = G\backslash\{a_1, a_2, , a_4\}$ and $\sigma = \sigma'(a_1a_4a_2)$.

Finally, if $T = a_1a_2a_3$, we put $G' = G\backslash\{a_1, a_2, a_3, x\}$ and $\sigma = \sigma'(xa_2a_1a_3)$.

In all the cases above $\sigma'$ is an embedding of $G'$ whose all cycles are of length 3 or 4. The existence of $\sigma'$ follows, by induction, from the choice of $G'$. It is easy to see that in all the cases $\sigma$ is an embedding of $G$ with one more cycle than $\sigma'$ and the lengths of all cycles are 3 or 4. ∎

THEOREM 2.7. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n - 2$, then there exists an embedding $\sigma$ of $G$ such that $\sigma$ is a cycle of length $n$.*

Proof. It is easy to see that the theorem holds for $n = 3, 4, 5$. Assume that it is true for $n$ and let $G$ be a graph of order $n + 1$ and of size at most $n - 1$. We shall distinguish two cases.

Case 1: $G$ has an end-vertex. Denote it by $x$ and consider the graph $G' = G\backslash\{x\}$. By induction, there exists a cyclic permutation $\sigma'$ of $V(G')$ that is an embedding of $G'$. Let $\sigma' = (a_1a_2 \ldots a_n)$. Without loss of generality we may assume that $a_1x$ belongs to $E(G)$. Observe that at least one of the edges $a_1a_2$ or $a_1a_n$ does not belong to $E(G)$. Suppose that $a_1a_2 \in E(G)$. Then $a_1a_n \notin E(G)$ and it is easy to see that the cyclic permutation on $V(G)$ defined by $\sigma = (a_1xa_2 \ldots a_n)$ is an embedding of $G$. If $a_1a_n \in E(G)$ then we put $\sigma = (a_1a_2 \ldots a_nx)$.

Case 2: $G$ contains no end-vertices. Then $G$ has at least two isolated vertices $x, y$. Denote by $z$ a vertex of $G$ with $d(z) \geq 2$ and consider the graph $G' = G\backslash\{x, z\}$. Denote by $\sigma'$ an embedding of $G'$ and let $\sigma' = (a_1a_2 \ldots a_{n-1})$. Without loss of generality we may assume that $a_1 = y$, i.e. $a_1$ is an isolated vertex of $G$. It is easy to see that the cyclic permutation on $V(G)$ defined by $\sigma = (a_1a_2 \ldots a_{n-1}xz)$ is an embedding of $G$. ∎

Remark. Observe that the above theorem gives a new proof of Theorem 2.1 about the existence of an embedding. Moreover, it provides a very simple algorithm for finding such an embedding.

As we have seen, if $e(G) = n - 1$ then there are graphs that are not embeddable and even in the case where a graph is embeddable, a fixed-point embedding does not necessarily exist. However, if we assume in addition that $G$ is a tree, we have the following result (cf. [80]), which, on the other hand, is also an improvement of Straight's result mentioned on page 10 (see also [24] for some related results).

Theorem 2.8. *Let $T$ be a tree of order $n$. If $T \neq S_n$ then there exists an embedding $\sigma$ of $G$ such that $\sigma$ is a cycle of length $n$.*

Proof. We shall define an algorithm for finding the embedding permutation.

Since $T \neq S_n$, the tree $T$ contains a path of length three. Let us label the vertices of this path by 1, 2, 3, 4 and denote it by $T_4$.

For each $k$, $4 < k \leq n$, label by "$k$" a vertex connected by an edge with $T_{k-1}$ which has not yet been labeled and denote by $T_k$ the tree induced by $\{1, 2, 3, 4, \ldots, k\}$.

Then the required permutation can be defined as follows:

For $k = 4$ it is given by the cyclic permutation (1342). For $k > 4$, let $\sigma' = (a_1 a_2 \ldots \ldots a_{k-1})$ be a cyclic permutation that is an embedding of $T_{k-1}$. Suppose that the vertex $k$ is adjacent to $a_i$ on $T_{k-1}$.

If the edge $a_{i-1}a_i$ does not belong to $E(T_{k-1})$, then we insert the vertex $k$ between $a_i$ and $a_{i+1}$, i.e. the permutation is defined by $\sigma = (a_1 a_2 \ldots a_i k a_{i+1} \ldots a_{k-1})$.

If the edge $a_{i-1}a_i$ belongs to $E(T_{k-1})$, then the permutation is given by $\sigma = (a_1 a_2 \ldots \ldots a_{i-1} k\ a_i \ldots a_{k-1})$.

By an argument similar to the previous theorem, it is easy to see that at each step of the above algorithm, the resulting cyclic permutation is an embedding of $T_k$. ∎

Figure 2.7 gives a "geometric" illustration of the above construction for the tree $S_6''$.

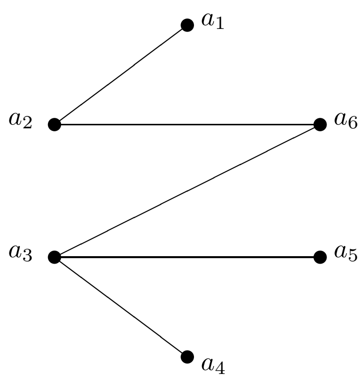



Fig. 2.7. Packing is given by the rotation $(a_1 a_2 \ldots a_6)$

As we have mentioned at the beginning of this section, there are $(n, n-1)$ graphs that are embeddable in their complements, but cannot be embedded without fixed vertices. Two simple examples are $K_{1,2} \cup C_3$ and $K_{1,3} \cup C_3$. It is interesting to note that all other $(n, n-1)$ graphs that are contained in their complements can be embedded without fixed vertices. More precisely, we have the following theorem (see [59]), which we state without proof.

THEOREM 2.9. *Let $G = (V, E)$ be a graph of order $n$ with $|E(G)| \leq n - 1$ and such that*

(a) *$G$ is not an exceptional graph of Theorem* 2.2,
(b) *$G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$.*

*Then there exists a fixed-point-free embedding of $G$.* ∎

**2.4. Graphs without small cycles.** The theorems about fixed-point-free embeddings considered in the previous section can be used to obtain a characterization of graphs of order and size equal to $n$ that are not embeddable. The result stated below without proof (its proof can be found in [31]) is the starting point for the main subject of this section, described in Conjecture 2.11.

THEOREM 2.10. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| = n$ then either $G$ is embeddable or $G$ is isomorphic to one of the graphs of Figure* 2.8. ∎

In [31] the authors remarked that all nonembeddable graphs with $n$ vertices and no more than $n$ edges (that is, the exceptional graphs of Theorems 2.2 and 2.10) are either stars or contain $K_3$ or $C_4$ as subgraphs. For this reason they have conjectured that

CONJECTURE 2.11. *Each nonstar graph which contains no cycles of length* 3 *or* 4 *as subgraphs is always embeddable.*

This conjecture, if true, would nicely fit with other characterization theorems which specify that all graphs, except a family of forbidden graphs, satisfy a given property or are of a given type.

The following theorem, proved in [31], provides some evidence that the above conjecture might hold.

THEOREM 2.12. *If a graph $G$ with $n$ vertices is not a star, contains no more than $(6/5)n - 2$ edges, and has no cycles of length* 3 *or* 4 *as subgraphs, then $G$ is embeddable.* ∎

Our purpose in this section is to prove the following

THEOREM 2.13. *If a graph $G$ is not a star and contains no cycles of length* 3, 4, 5, 6 *or* 7 *as subgraphs, then $G$ is embeddable.*

The proof of this theorem is given at the end of this section. We begin with lemmas concerning some special cases. We shall need some additional definitions and notation.

Let $G$ be a connected graph with $\operatorname{diam}(G) = d$ and let $A$ be a subset of $\{0, 1, 2, \ldots, d\}$. A permutation $\sigma$ on $V(G)$ is said to belong to the *class* $\mathcal{P}(G, A)$ iff for every $x \in V(G)$, $\operatorname{dist}_G(x, \sigma(x)) \in A$.

Note that if $0 \notin A$ and $\sigma \in \mathcal{P}(G, A)$, then $\sigma$ has no fixed points.

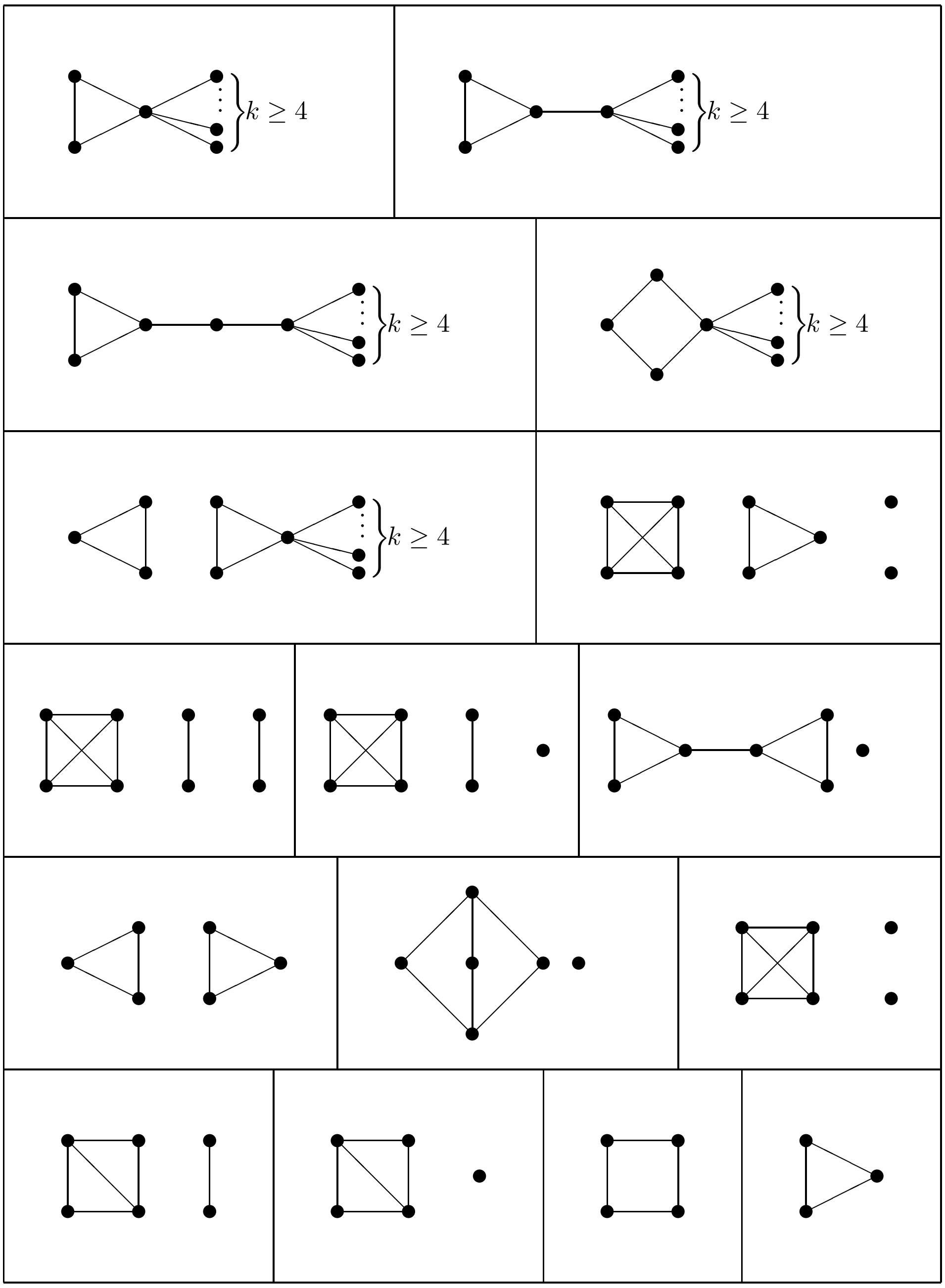



Fig. 2.8. Non-embeddable $(n, n)$ graphs

LEMMA 2.14. *Let $P_n$ be a path of order $n$ with $n > 3$. Then there exists an embedding of $P_n$ belonging to $\mathcal{P}(P_n, \{1,2\})$.*

Proof. The fact that our lemma is true in the cases where $n = 4, 5, 6, 7$ is easy to prove and we leave this to the reader.

For $n > 7$ observe that there exists an edge $e$ of $P_n$ such that $P_n - e$ has two components $P_i, P_j$ with $i, j > 3$. Then, by induction hypothesis, there exist embeddings $\sigma_i$ of $P_i$ and $\sigma_j$ of $P_j$ such that $\sigma_i \in \mathcal{P}(P_i, \{1,2\})$ and $\sigma_j \in \mathcal{P}(P_j, \{1,2\})$. It is easy to see that the permutation $\sigma$ of $V(P_n)$ that extends both $\sigma_i$ and $\sigma_j$ belongs to $\mathcal{P}(P_n, \{1,2\})$ and, since $\sigma_i, \sigma_j$ have no fixed points, $\sigma$ is also an embedding of $P_n$. ■

LEMMA 2.15. *Let $G'$ be a connected graph and $a \in V(G')$. Let $G$ be a graph obtained from $G'$ by adding $k + k'$ new vertices $x_1, \ldots, x_k, y_1, \ldots, y_{k'}$ and $k + k'$ new edges constituting two paths $ax_1 \ldots x_k$ and $ay_1 \ldots y_{k'}$ of lengths $k$ and $k'$, respectively, and having the vertex $a$ as the common end vertex, $1 \le k, k' \le 3$. Suppose there exists an embedding $\sigma'$ of $G'$ such that $\sigma' \in \mathcal{P}(G', \{1,2,3\})$. Then there exists an embedding $\sigma$ of $G$ such that $\sigma \in \mathcal{P}(G, \{1,2,3\})$.*

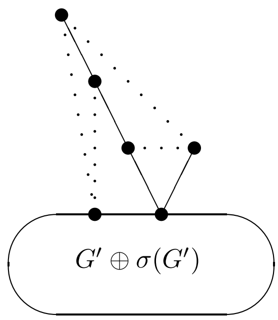



Fig. 2.9. Case $k = 3$, $k' = 1$

Proof. The proof is by the case-by-case examination. We give two examples below, the third is given in Figure 2.9 and the other cases are left to the reader.

If $k = k' = 1$ we define $\sigma$ as follows: $\sigma(x_1) = y_1$, $\sigma(y_1) = x_1$ and $\sigma(x) = \sigma'(x)$ for $x \in V(G')$. Since $\mathrm{dist}_G(x_1, y_1) = 2$ we have $\sigma \in \mathcal{P}(G, \{1,2,3\})$. On the other hand, $\sigma$ is an embedding of $G$ because $\sigma'(a)$ is different from $a$.

If $k = k' = 3$, then we put $\sigma(x_1) = x_3$, $\sigma(x_2) = y_1$, $\sigma(x_3) = x_2$, $\sigma(y_1) = y_3$, $\sigma(y_2) = x_1$, $\sigma(y_3) = y_2$ and $\sigma(x) = \sigma'(x)$ for $x \in V(G')$. ■

LEMMA 2.16. *If $T$ is a tree of order $n$ and $T$ is not a star, then there is an embedding $\sigma$ of $T$ such that $\sigma \in \mathcal{P}(G, \{1,2,3\})$.*

Proof. For $n < 4$ each tree is a star so suppose that $n \ge 4$. For $n = 4$ there is only one tree which is not a star, namely the path $P_4$, so our lemma follows from Lemma 2.14. Suppose now that it holds for $n' < n$.

If $\mathrm{diam}(T) \ge 7$ then there exists an edge $e$ such that $T - e$ has two components $T', T''$ that are not stars. The embedding of $T$ belonging to $\mathcal{P}(T, \{1,2,3\})$ can be easily obtained from the corresponding embeddings of $T'$ and $T''$ since they do not have any fixed points.

So we may assume that $\mathrm{diam}(T) \leq 6$. Observe that if $\mathrm{diam}(T) = 5$ or 6 then either Lemma 2.14 or Lemma 2.15 can be always applied.

Consider now the case where $\mathrm{diam}(T) = 4$. Let $x_1, \ldots, x_5$ be the longest path of $T$. Observe that if there were a vertex of $T - \{x_1, \ldots, x_5\}$ adjacent to $x_2$ or $x_4$, then we could apply Lemma 2.15. It is easy to see that there are only two cases where neither Lemma 2.14 nor Lemma 2.15 can be used. The first one is the graph obtained from the path $x_1, \ldots, x_5$ by adding two new vertices $y_1$ and $y_2$ and two edges $x_3y_1$ and $y_1y_2$. The second graph is obtained from the path $x_1, \ldots, x_5$ by adding one new vertex $y$ connected by an edge with $x_3$. In the first case the permutation $\sigma$ can be defined as follows: $\sigma(x_1) = x_3$, $\sigma(x_2) = x_1$, $\sigma(x_3) = x_5$, $\sigma(x_4) = y_1$, $\sigma(x_5) = x_2$, $\sigma(y_1) = y_2$, $\sigma(y_2) = x_4$.

In the second case we put: $\sigma(x_1) = x_3$, $\sigma(x_2) = x_1$, $\sigma(x_3) = x_5$, $\sigma(x_4) = y$, $\sigma(x_5) = x_4$, $\sigma(y) = x_2$.

Finally, observe that in the case where $\mathrm{diam}(T) = 3$, every embedding which has no fixed points has the required property. Thus, we can apply, for example, Theorem 2.8. ■

Proof of Theorem 2.13. Observe first that without loss of generality one may assume that $G$ is connected. For, suppose that the graph $G$ is not connected and denote by $r$ the number of components of $G$, $r \geq 2$. Add $r-1$ edges to join distinct components. An embedding of the connected graph obtained is also an embedding of $G$.

Therefore, we may assume that $G$ is a connected graph. We may also assume that $n > 8$. In this case we shall prove that there exists an embedding $\sigma$ of $G$ such that $\sigma \in \mathcal{P}(G, \{1, 2, 3\})$. By Lemma 2.16 we may assume that $G$ is not a tree.

Let $abcd$ be a path of length 3 contained in a cycle of $G$. Suppose that $G' = G - \{a, b, c, d\}$ has a component $S$ that is a star and let $xy$ be edge of $G$ connecting the path $abcd$ and $S$, $x \in \{a, b, c, d\}$. Note that there is only one such edge since $G$ does not contain any cycle of length less than eight. We can assume that either $S = K_1$ or $S = P_2 = yy_1$ or $S = P_3 = yy_1y_2$, for otherwise we could use Lemma 2.15. This implies, by the same argument, that we cannot find two stars that are components of $G'$ and are connected to the same vertex on the path $abcd$.

If both vertices $b$ and $c$ are connected with two stars $S'$ and $S''$ (the components of $G'$) then we replace the path $abcd$ by the path $P'$ with the vertex set $V(S') \cup \{b, c\} \cup V(S'')$. Observe that no component of the graph $G'' = G - P'$ is a star since $a$ and $d$ are connected by a path.

If only the vertex $b$ is connected by an edge with a star component $S'$ of $G'$, we replace the path $abcd$ by a path with the vertex set $V(S') \cup \{b, c, d\} \cup V(S'')$, where $S''$ denotes a star component of $G'$ connected with $d$ (if it exists). We proceed similarly if there is no star that is a component of $G'$ connected by an edge with $\{b, c\}$.

Thus we have shown that it is always possible to choose a path $P$ of a length greater than 2 in such a way that $G' = G - P$ has no component which is a star.

Denote by $G_i$ the components of $G'$ and by $\sigma_i$ the embeddings of $G_i$ belonging to $\mathcal{P}(G_i, \{1, 2, 3\})$, $i = 1, \ldots, k$. Let $\sigma_0$ be an embedding of $P$ belonging to $\mathcal{P}(P, \{1, 2\})$ (cf. Lemma 2.14). We put $\sigma(x) = \sigma_i(x)$ for $x \in V(G_i)$ and $\sigma(x) = \sigma_0(x)$ for $x \in V(P)$. Clearly, $\sigma \in \mathcal{P}(G, \{1, 2, 3\})$.

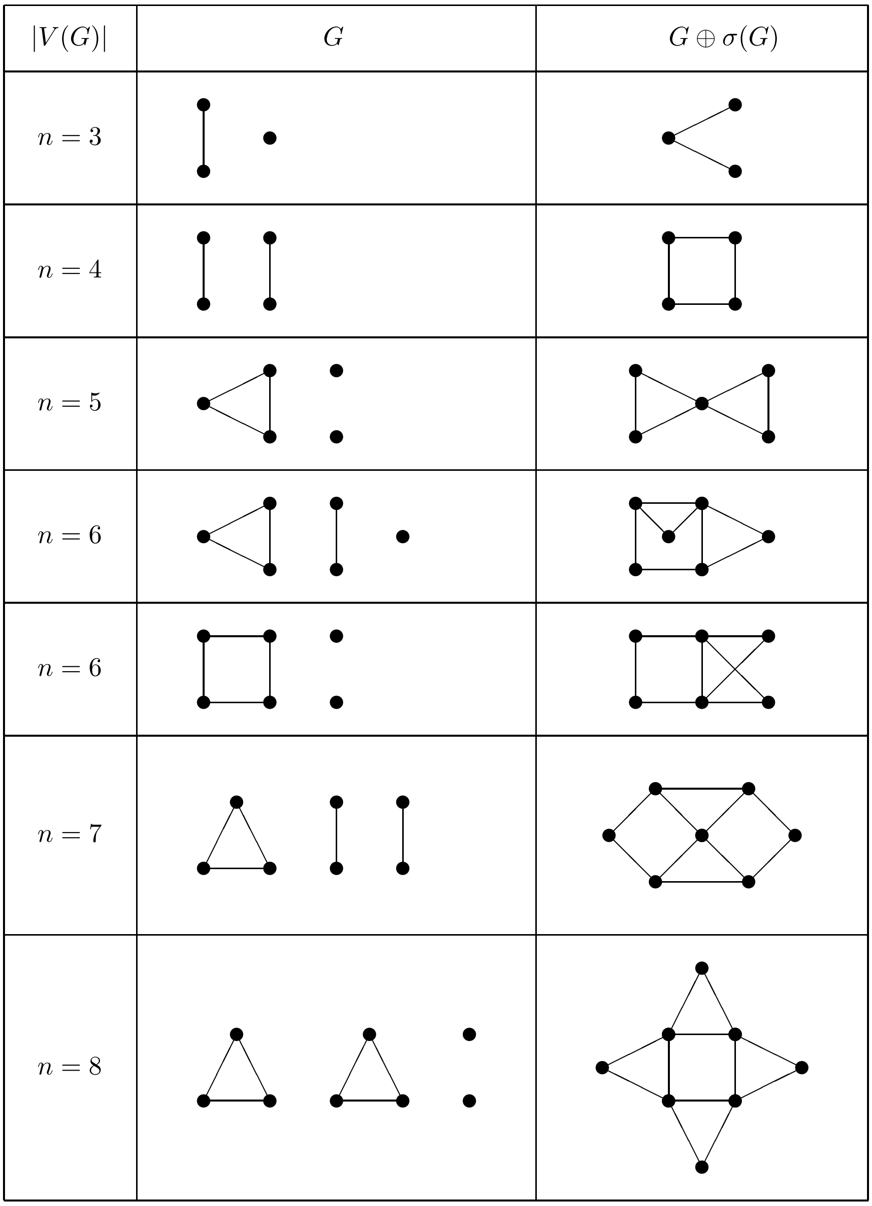



Fig. 2.10. Uniquely embeddable $(n, n-2)$ graphs

Suppose that $\sigma$ is not an embedding of the graph $G$. Thus, by the definition of $\sigma$, there exist two edges of $G$, say $xy$ and $x'y'$, such that $\sigma(x)\sigma(y) = x'y'$ and $x, x' \in V(P)$, $y, y' \in V(G_i)$ for some $i$.

Since $\sigma_0 \in \mathcal{P}(P, \{1,2\})$ and $\sigma_i \in \mathcal{P}(G_i, \{1,2,3\})$, there exists a path of length $\leq 2$ from $x$ to $x'$ contained in $P$ and a path of length $\leq 3$ from $y$ to $y'$ contained in $G_i$. These paths together with the edges $xx'$ and $yy'$ constitute a cycle of length less than or equal to 7, which is a contradiction. ■

We finish this section by stating without proof the following theorem of Brandt [11] which evidently improves Theorem 2.13.

THEOREM 2.17. *If a graph $G$ is not a star and contains no cycles of length* 3, 4, 5 *or* 6 *as subgraphs*, *then $G$ is embeddable.* ■

**2.5. Uniquely embeddable graphs.** We finish this chapter by giving, without proof, a result on the problem of the uniqueness of the embedding. First, we have to make precise what do we mean by *distinct* embeddings.

Let $\sigma$ be an embedding of the graph $G = (V, E)$. We denote by $\sigma(G)$ the graph with the vertex set $V$ and the edge set $\sigma^*(E)$, where the map $\sigma^*$ is induced by $\sigma$. Since, by the definition of an embedding, the sets $E$ and $\sigma^*(E)$ are disjoint, we may form the graph $G \oplus \sigma(G)$.

Two embeddings $\sigma_1$, $\sigma_2$ of a graph $G$ are said to be *distinct* if the graphs $G \oplus \sigma_1(G)$ and $G \oplus \sigma_2(G)$ are not isomorphic. A graph $G$ is called *uniquely embeddable* if, for all embeddings $\sigma$ of $G$, all graphs $G \oplus \sigma(G)$ are isomorphic.

The following theorem characterizes all $(n, n-2)$ graphs that are uniquely embeddable.

THEOREM 2.18. *Let $G$ be a graph of order $n$ and size $e(G) = n - 2$. Then either $G$ is not uniquely embeddable or $G$ is isomorphic to one of the following seven graphs* (*cf. Fig.* 2.10): $K_2 \cup K_1$, $2K_2$, $K_3 \cup 2K_1$, $K_3 \cup K_2 \cup K_1$, $C_4 \cup 2K_1$, $K_3 \cup 2K_2$, $2K_3 \cup 2K_1$. ■

The proof is given in [83]. The main tool is Theorem 2.7 on the existence of an embedding which is a cyclic permutation.

## 3. Packing of two graphs

**3.1. Packing of two graphs of small size.** The following theorem is an immediate generalization of Theorem 2.1 to the case of packing of two graphs. It was first proved in [57]. The proof presented here is a slight modification of the original proof.

THEOREM 3.1. *Let $G$ and $H$ be two graphs of order $n$. If $e(G) \le n-2$ and $e(H) \le n-2$, then $G$ and $H$ are packable into the complete graph $K_n$, i.e. there is a packing of $G$ and $H$.*

Proof. We use induction on $n$. The proof is straightforward for $n \le 5$. Without loss of generality we may assume that $e(G) = n - 2$ and $e(H) = n - 2$.

The proof is split into several cases.

Case (a): Both $G$ and $H$ have two independent end-edges. We remove two end-vertices in each graph and proceed by induction analogously to Case 1 of Theorem 2.1.

Case (b): Both $G$ and $H$ have at least one isolated vertex. Clearly, both $G$ and $H$ have at least one vertex of valency at least two. Then in each graph we remove an isolated vertex and one vertex of valency at least two and proceed by induction analogously to Case 2 of Theorem 2.1.

C a s e (c): $G$ has two independent end-edges, at least one of them is not an isolated edge and $H$ has at least one isolated vertex. Denote by $x$, $y$ two adjacent vertices of $G$ such that $d_G(x) = 1$ and $d_G(y) \geq 2$. Let $u$ be an isolated vertex of $H$ and let $v$ be a vertex of valency at least two (the existence of such a vertex is implied by the fact that $e(H) = n - 2$ and $n \geq 5$). Consider the graphs $G' = G \setminus \{x, y\}$ and $H' = H \setminus \{u, v\}$. The induction hypothesis can be applied to $G'$ and $H'$ and the extension of the packing of $G'$ and $H'$ to the packing of $G$ and $H$ is straightforward.

C a s e (d): $G$ has two isolated edges. We can assume that (a), (b) and (c) do not hold. This implies, in particular, that the graph $H$ has an isolated vertex $w$. Now, if we can find two nonadjacent vertices $u, v$ in $H$ which cover at least three edges of $H$ then we are done, which is immediately seen from Figure 3.1. For instance, let $u$ be a vertex of maximum degree in $H$ and let $v$ be any vertex of $H$, different from $w$ and not adjacent to $u$ with $d_H(u) + d_H(v) \geq 3$. It is easy to see that we always get the required configuration except for the cases where $H = K_1 \cup S_{n-1}$ or $n = 5$ and $H = 2K_1 \cup K_3$. These cases are left to the reader. ∎

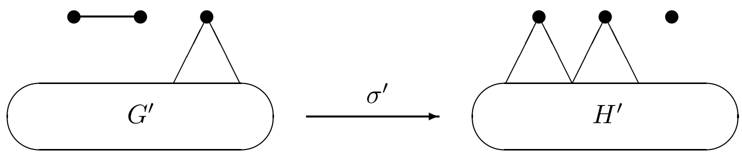



Fig. 3.1. Case (d) of the proof

We now turn our attention to the problem of packing two trees. The theorem below was proved in [42]. We shall need the following lemma (also proved in [42]).

LEMMA 3.2. *If $T$ is a tree of order $n$ other than the star $S_n$, then $T$ and $S'_n$ can be packed into $K_n$.*

P r o o f. Denote by $c$ the centre of $S'_n$, by $b$ the vertex of valency two, and by $a$ the vertex adjacent to $b$.

Let $v$ be an end-vertex of $T$ and let $w$ be the vertex of $T$ adjacent to $v$. Since $T$ is not a star, there is a vertex $x$ of $T$ that is not adjacent to $w$.

Observe that $S'_n$ can be placed with $T$ by putting $c$ on $v$, $b$ on $x$, $a$ on $w$ and arranging arbitrarily the remaining vertices of $S'_n$. ∎

The following theorem is a generalization of the result concerning the embedding of a tree.

THEOREM 3.3. *Any two trees of order $n$, neither of which is a star, can be packed into $K_n$.*

P r o o f. The proof is by induction on $n$. For $n = 4$ there is only one tree that is not a star, hence the packing is in fact an embedding. The situation is similar for $n = 5$ except for the pair $\{S'_5, P_5\}$, to which we can apply Lemma 3.2 (cf. also Figure 3.2). Hence, assume that $n \geq 6$ and let $T$ and $U$ be two nonstar trees of order $n$. By Lemma 3.2 we

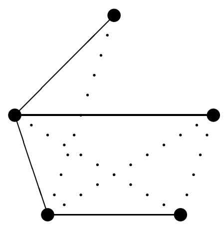


Fig. 3.2. Packing of $S'_5$ and $P_5$

may assume that neither $T$ nor $U$ is isomorphic to $S'_n$, and so each has two end-vertices at distance at least 3 such that after removing them we do not get a star.

Let $t_1$ and $t_2$ be such end-vertices of $T$. We choose $u_1$ and $u_2$ in $U$ in a similar way. By the induction hypothesis, $T' = T \setminus \{t_1, t_2\}$ and $U' = U \setminus \{u_1, u_2\}$ can be packed into $K_{n-2}$. It is now easy to complete this packing and to obtain a packing of $T$ and $U$ into $K_n$. ■

The last theorem has been improved by Slater, Teo and Yap [63]:

THEOREM 3.4. *Let $T$ be a tree of order $n$, $n \geq 5$, and let $G$ be an $(n, n-1)$ graph. If neither $T$ nor $G$ is a star, then there is a packing of $T$ and $G$.* ■

Remark. A further improvement of the last result which provides a complete characterization of packing of a tree of order $n$ with an $(n, n)$ graph is given in the last section of this chapter. We shall also give there a complete characterization of packing of two $(n, n-1)$ graphs, a result which improves Theorem 3.1, as well as Theorem 3.4, in a different way.

**3.2. Packing an undense and a dense graph.** For the sake of completeness, we formulate here some examples of packing theorems which hold for $n$ sufficiently large. The first two of them concern the situation where one of the graphs is supposed to be *undense* (that is, it is a graph of order $n$ with less than $n/2$ edges). Then the second graph may be relatively "dense".

The main references for this section are [6], [8] as well as recent Brandt's papers [13], [14].

Recall that we denote by $T_r(n)$ the graph of Turán, that is, the complete $r$-partite graph with colour classes as close as possible (cf. page 8) and $t_r(n) = e(T_r(n))$.

The next theorem was proved in [8].

THEOREM 3.5. *Let $k$ be a fixed natural number and let $s$ be given by $\binom{s}{2} \leq k < \binom{s+1}{2}$. If $e(G_1) = k$ and $e(G_2) \leq \binom{n}{2} - t_{s-1}(n) - 1$, then for sufficiently large $n$ there is a packing of $G_1$ and $G_2$. The example $G_1 = K_s \cup \overline{K}_{n-s}$, $G_2 = \overline{T}_{s-1}(n)$ shows that the result is best possible.* ■

Remark. As observed in [13], the restriction $m \geq 3$ was not required in [8] but for $m = 2$ the statement is obviously false.

Also in [8] the authors stated the following conjecture:

Conjecture 3.6. *Let $0 < \alpha < 1/2$ and $0 < c < 1/\sqrt{8}$ be fixed. If $n$ is sufficiently large and $e(G_1) \le \alpha n$, $e(G_2) \le (c/\sqrt{\alpha})n^{3/2}$, then there is a packing of $G_1$ and $G_2$.*

Remark. If $G_1 = K_{t+1} \cup \overline{K}_{n-t-1}$ and $G_2 = tK_{n/t}$, then clearly there is no packing of $G_1$ and $G_2$. This example shows that $c$ cannot be greater than $1/\sqrt{8}$.

This conjecture was proved by Brandt in [13]. It is a consequence of the following theorem, which is formulated here in terms of packing.

Theorem 3.7. *Suppose that $n \ge 517$ and $3 \le m < n/2$ and let $s$ be the integer satisfying $\binom{s}{2} \le m < \binom{s+1}{2}$. If $G_1$ and $G_2$ are graphs of order $n$ with $e(G_1) \le m$ and $e(G_2) < \binom{n}{2} - \max\left\{t_{s-1}(n), \binom{2m-1}{2}\right\}$, then there is a packing of $G_1$ and $G_2$.* ∎

We finish this section by mentioning one more result which holds for $n$ sufficiently large. This result was obtained by Komlós, Sárközy and Szemerédi (see [46]). It proves a conjecture stated by Bollobás in [6].

Theorem 3.8. *Let $\Delta$ and $0 < c < 1/2$ be given. Then there exists a constant $n_0$ with the following properties: If $n \ge n_0$, $T$ is a tree of order $n$ and maximum degree $\Delta$, and $G$ is a graph of order $n$ and maximum degree not exceeding $cn$, then there is a packing of $T$ and $G$.* ∎

As remarked in [46], it is easy to see that the statement is not true for $c = 1/2$. For example, for every even $n$, if $G = K_{n/2,n/2}$, and $T$ is a path with $n$ vertices, then obviously there is no packing of $T$ and $G$.

### 3.3. Products of sizes and degrees.

In this section we shall consider some other conditions (not concerning the size) ensuring the packing of two graphs. We start with two theorems proved in [57].

Theorem 3.9. *If $e(G)e(H) < \binom{n}{2}$ then $G$ and $H$ are packable.*

Proof. We shall use a *probabilistic method*. Consider the probability space whose $n!$ points are all the possible bijections $\sigma$ from $V(G)$ to $V(H)$, each with probability $(n!)^{-1}$.

For any two edges $e \in E(G)$ and $f \in E(H)$ we denote by $\mathcal{A}_{ef}$ the event that $f$ is an image of $e$. In other words,

$$\mathcal{A}_{ef} = \{\sigma : V(G) \to V(H) \mid \sigma^*(e) = f\}.$$

Then

$$\mathrm{Prob}(\mathcal{A}_{ef}) = \frac{2(n-2)!}{n!} = \binom{n}{2}^{-1}.$$

Let $\mathcal{A} = \bigcup \mathcal{A}_{ef}$, where the union is taken over all $(e,f) \in E(G) \times E(H)$. Then

$$\mathrm{Prob}(\mathcal{A}) = \mathrm{Prob}\left(\bigcup \mathcal{A}_{ef}\right) \le \sum \mathrm{Prob}(\mathcal{A}_{ef}) = e(G)e(H)\binom{n}{2}^{-1} < 1.$$

Hence for a certain $\sigma$ the event $\mathcal{A}$ does not hold. That is, there exists a permutation $\sigma$ which is a packing of $G$ and $H$. ∎

Remark. The above theorem is, in a certain sense, best possible. For, let $G$ be a star with $n-1$ edges for even $n$ and let $H$ be a matching with $n/2$ edges. Then clearly $G$ and $H$ are not packable.

THEOREM 3.10. *Let $G$ and $H$ be graphs of order $n$. If $2\Delta(G)\Delta(H) < n$, then $G$ and $H$ are packable.*

Proof. Let $\sigma : V(G) \to V(H)$ be a bijection that minimizes $|\sigma(E(G)) \cap E(H)|$. Assume that $ac \in \sigma(E(G)) \cap E(H)$. We shall arrive at a contradiction by finding $\sigma'$ with a smaller intersection. We examine those $b \in V(G)$ such that one of the conditions holds:

(i) $b = a$,
(ii) the edge $ab$ belongs to $\sigma(E(G)) \cap E(H)$,
(iii) for some $x \in V(H)$, $ax \in \sigma(E(G))$, $xb \in E(H)$,
(iv) for some $y \in V(H)$, $ay \in \sigma(E(H))$, $yb \in E(G)$.

There are at most $\Delta(G)\Delta(H)$ vertices $b$ satisfying (iii), and similarly for (iv). More precisely, if $t$ is the number of $b$'s satisfying (ii), then at most $\Delta(G)\Delta(H) - t$ satisfy (iii), and similarly for (iv). Hence at most $1 + t + 2(\Delta(G)\Delta(H) - t) \le 2\Delta(G)\Delta(H)$ vertices satisfy one of the above conditions. Fix $b \in V(H)$ satisfying *none* of (i)–(iv). Define $\sigma'$ by

$$\sigma'(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \neq a, b, \\ b & \text{if } \sigma(i) = a, \\ a & \text{if } \sigma(i) = b, \end{cases}$$

that is, interchange $a$ and $b$ in the map $\sigma'$. Let $xy \in \sigma'(E(G)) \cap E(H)$. If $\{x, y\} \cap \{a, b\} = \emptyset$, this means that $xy$ was in $\sigma(E(G)) \cap E(H)$. But no edge of the form $az$ or $bz$ can be in $\sigma'(E(G)) \cap E(H)$. For example, if $az \in \sigma'(E(G)) \cap E(H)$, then $az \in E(H)$, $bz \in \sigma(E(G))$, but then $b$ would satisfy (iv). The other cases are similar. Hence $|\sigma'(E(G)) \cap E(H)| < |\sigma(E(G)) \cap E(H)|$, completing the proof of the theorem. ∎

Remark. A generalization of the above theorem can be found in [13], [14].

Consider now the following example ([8]). Let $d_1 \le d_2 < n$ be integers and suppose that $n \le (d_1 + 1)d_2$. Given natural numbers $d_1$ and $d_2$, put $n_0 = (d_1 + 1)(d_2 + 1) - 2$ and let $G_1 = d_2 K_{d_1+1} \cup K_{d_1-1}$ and $G_2 = d_1 K_{d_2+1} \cup K_{d_2-1}$. Then $v(G_1) = v(G_2) = n_0$, $\Delta(G_1) = d_1$, $\Delta(G_2) = d_2$ and there is no packing of $G_1$ and $G_2$.

Motivated by this example, Bollobás and Eldridge made the following conjecture (see [8]), stated also, independently, by Catlin (cf. [20]).

CONJECTURE 3.11. *If $G$ and $H$ are graphs of order $n$ and*

$$(\Delta(G) + 1)(\Delta(H) + 1) \le n + 1,$$

*then there is a packing of $G$ and $H$.*

The example given above shows that the conjecture, if true, is the best possible. Here is another example given by Catlin in [20].

Let $g$ and $h$ be two integers such that $(g + 1)(h + 1) = n + 2$. We put $G = (g - 1)K_{h+1} \cup K_{h,h}$ and $H = gK_{h+1} \cup K_{h-1}$. We clearly have $\Delta(G) = g$ and $\Delta(H) = h$. It is easy to see that these graphs are not packable.

As mentioned in [6], the above conjecture holds for $\Delta(G) = 1$. The case $\Delta(G) = 2$ was proved by Aigner and Brandt in [1]. We state below this result in a subgraph form.

THEOREM 3.12. *Let $H$ be a graph of order $n$ with $\delta(H) \ge (2n-1)/3$. Then $H$ contains each graph $G$ of order less than or equal to $n$ with $\Delta(G) = 2$.* ∎

This result improves the following theorem of Corrádi and Hajnal [25].

Theorem 3.13. *Let $H$ be a graph of order $n$ with $\delta(H) \geq (2n-1)/3$. Then $H$ contains the graph $G$ composed of $\lfloor n/3 \rfloor$ triangles $C_3$.* ∎

Recently, Fan and Kierstead [30] proved that

Theorem 3.14. *Let $H$ be a graph on $n$ vertices with $\delta(H) \geq (2n-1)/3$. Then $H$ contains a hamiltonian square-path, that is, a graph obtained from a hamiltonian path by joining every pair of vertices of distance two in the path.* ∎

Clearly, this theorem improves Theorem 3.12 because every graph $G$ with $\Delta(G) = 2$ is contained in a hamiltonian square-path. Let us mention that Posá and Seymour conjectured that if $G$ is a graph on $n$ vertices with $\delta(G) \geq 2n/3$, then $G$ contains a hamiltonian square-cycle. In 1973 Seymour introduced a more general conjecture that if $\delta(G) \geq kn/(k+1)$, then $G$ contains the $k$th power of a hamiltonian cycle, that is, a graph obtained from a hamiltonian cycle by joining every pair of vertices of distance $k$ in the cycle (cf. [30]).

Note also that the following result of Hajnal and Szemerédi [39] is in fact a special case of Conjecture 3.11.

Theorem 3.15. *Let $H$ be a graph of order $n$ with $\delta(H) \geq (hn-1)/(h+1)$, $h \geq 2$. Then $H$ contains the graph $G$ composed of $\lfloor n/(h+1) \rfloor$ complete graphs $K_{h+1}$.* ∎

Observe that Theorem 3.12 gives a sufficient condition for a graph to have as a subgraph each graph composed of edge-disjoint cycles. It nicely agrees with the following classical theorem of Dirac on the existence of hamiltonian cycle.

Theorem 3.16. *Let $H$ be a graph of order $n$ with $\delta(H) \geq n/2$. Then $H$ contains a hamiltonian cycle $C_n$.* ∎

Finally, we give a theorem which, in a certain sense, is between Dirac's theorem and Theorem 3.12.

Theorem 3.17. *Let $H$ be a graph of order $n$ with $\delta(H) \geq n/2$. Then either $H$ is pancyclic (that is, it contains all cycles $C_p$ with $3 \leq p \leq n$) or $H$ is the complete bipartite graph $K_{n/2,n/2}$.* ∎

Note that by Theorem 3.16 the above theorem is an immediate consequence of the following well-known result of Bondy (cf. also [20]).

Theorem 3.18. *Let $H$ be a graph of order $n$. If $H$ is hamiltonian and $|E(H)| \geq n^2/4$, then either $H$ is pancyclic or $H$ is the complete bipartite graph $K_{n/2,n/2}$.* ∎

We refer the reader to the survey paper [47], where a variety of related results, some quite recent, is discussed.

**3.4. Sum of sizes.** We shall consider conditions on the sum of sizes of two graphs $G$ and $H$ that ensure the possibility of packing of $G$ and $H$.

Suppose that $G$ and $H$ are graphs of order $n$. If $\Delta(G) = n-1$ and $\delta(H) \geq 1$ (or $\Delta(H) = n-1$ and $\delta(G) \geq 1$), then obviously there is no packing of $G$ and $H$.

We begin with a result conjectured by Milner and Welsh and proved by Sauer and Spencer in [57] as a corollary to our Theorems 3.1 and 3.9. We present a direct proof given in [8].

THEOREM 3.19. *Suppose that $G$ and $H$ are two graphs of order $n$. If $e(G)+e(H) \le \lfloor 3(n-1)/2 \rfloor$, then there is a packing of $G$ and $H$.*

Proof. The result is obvious if $n \le 4$, so we consider $n \ge 5$ and assume that the theorem is true for $n' < n$. Without loss of generality we may assume that $e(G)+e(H) = \lfloor 3(n-1)/2 \rfloor$ and $e(G) \le e(H)$. If one of the graphs has an isolated vertex $x$ and the other has a vertex $y$ of degree at least 2, then we can put $x$ on $y$ and complete the packing by the induction hypothesis. So we may assume that this does not happen. We have $\Delta(H) \ge 2$, since otherwise

$$e(G)+e(H) \le \frac{n}{2}+\frac{n}{2} = n < \left\lfloor \frac{3(n-1)}{2} \right\rfloor .$$

Therefore, $G$ has no isolated vertices. Let $T$ be a component of $G$ with the smallest density. Then $T$ has density less than $3/4$, so $T$ is a path with two or three vertices. If $T = xy$, we may put $x$ on a vertex $u$ of the maximum degree in $H$, and $y$ on any vertex not adjacent to $u$. We can do this since

$$e(H) \le \left\lfloor \frac{3(n-1)}{2} \right\rfloor - \frac{n}{2} < n-1.$$

Then $T$ covers at least two edges and the packing can be completed by induction. If $T = xyz$, then $\Delta(G) \ge 2$, so $H$ has no isolated vertex. On the other hand,

$$e(H) \le \left\lfloor \frac{3(n-1)}{2} \right\rfloor - \frac{2n}{3} < n-1,$$

so $H$ has at least two components. Put $y$ on a vertex $u$ of the maximum degree in $H$ and $x$, $z$ on any vertices in another component. Then $T$ covers at least three edges and the packing can be completed by the induction hypothesis. ∎

Note that the example given in the remark after Theorem 3.9 proves that, also in this case, the theorem is best possible. However, if we assume in addition that neither $G$ nor $H$ contains a vertex of degree $n-1$, then the result can be considerably improved.

The following theorem is called the *Main Packing Theorem* ([8]).

THEOREM 3.20. *Suppose that $H$ and $G$ are graphs of order $n$, $\Delta(H) < n-1$, $\Delta(G) < n-1$, $e(G)+e(H) \le 2n-3$ and $\{H,G\}$ is not one of the following pairs (cf. Figure 3.3): $\{2K_2, K_1 \cup K_3\}$, $\{2K_1 \cup K_3, K_2 \cup K_3\}$, $\{3K_2, 2K_1 \cup K_4\}$, $\{3K_1 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, 3K_1 \cup K_4\}$, $\{4K_1 \cup K_4, K_2 \cup 2K_3\}$, $\{5K_1 \cup K_4, 3K_3\}$. Then there is a packing of $H$ and $G$.* ∎

We do not present the proof of this theorem because it is given in both main references [6] and [85]. The main idea of the proof is similar to that of the previous theorem. Observe that without loss of generality we may assume that $e(H)+e(G) = 2n-3$, $e(H) \le n-2$ and $e(G) \ge n-1$. Let $p$ be the order of the smallest tree-component $T$ (which may be $K_1$) of $H$. Then $2p \le n$ and $e(H) \ge n - n/p$. Hence $e(G) \le n + n/p - 3$. This is why

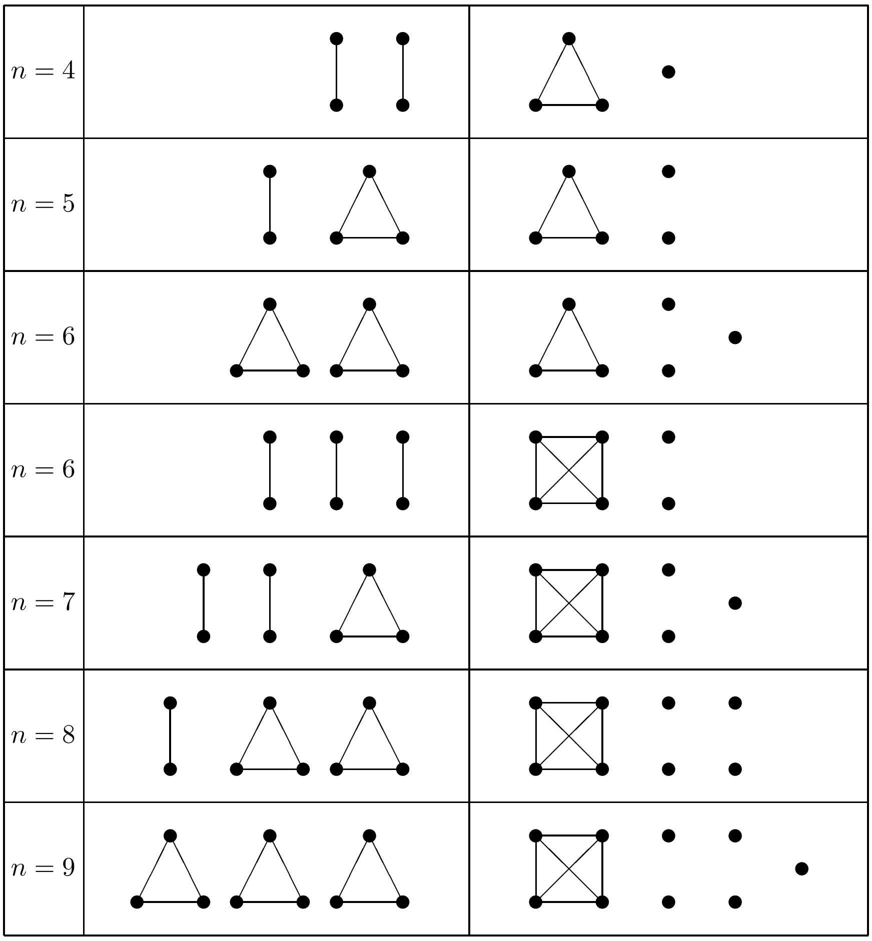



Fig. 3.3. Couples of non-packable graphs with the sum of sizes equal to $2n - 3$

the proof of the Main Packing Theorem depends on the following lemma [8] (see also [6] and [85]).

LEMMA 3.21. *Let $T$ be a tree of order $p$ and let $G$ be a graph of order $n$. Suppose that $2 \le 2p \le n$, $n \ge 5$, $\Delta(G) < n-1$ and $n-1 \le e(G) \le n+n/p-3$. Then there is a packing of $T$ and $G$ such that $T$ covers at least $p+1$ edges of $G$ and $\Delta(G-T) < n-p-1$.* ■

R e m a r k. Observe that for all exceptional pairs of graphs of the Main Packing Theorem (cf. Figure 3.3) nonexistence of a packing is clear. Moreover, in all the cases it directly follows from Turán's theorem (cf. page 8). Indeed, all left-hand side graphs in Figure 3.3 are in fact the complements of Turán's graphs $T_r(n)$ for $r = 2$ or 3 and $4 \le n \le 9$.

The following corollary can be considered as a “without exceptions” version of the Main Packing Theorem.

COROLLARY 3.22. *Suppose that $G$ and $H$ are graphs of order $n$ such that neither of the following conditions holds*:

(a) $\Delta(H) = n-1$ *and* $\delta(G) \geq 1$,
(b) $\Delta(G) = n-1$ *and* $\delta(H) \geq 1$.

*If $e(G) + e(H) \leq 2n-4$, then there is a packing of $G$ and $H$.* ∎

Note also that the Main Packing Theorem can be formulated as follows.

COROLLARY 3.23. *Suppose that $G$ and $H$ are graphs of order $n$ such that neither of the following conditions holds*:

(a) $\Delta(H) = n-1$ *and* $\delta(G) \geq 1$,
(b) $\Delta(G) = n-1$ *and* $\delta(H) \geq 1$.

*If $e(G) + e(H) \leq 2n-3$ and $\{G, H\}$ is not one of the forbidden pairs of Theorem* 3.20, *then there is a packing of $G$ and $H$.*

Proof. Suppose that $\Delta(G) \leq \Delta(H)$. If $\Delta(H) < n-1$, then the result follows from Theorem 3.20. If $\Delta(H) = n-1$ and $\delta(G) = 0$, then we can map an isolated vertex $x$ of $G$ to a vertex $y$ of $H$ such that $d(y) = n-1$. Then $e(G-x) + e(H-y) \leq n-2$ and one can easily find a packing of $G-x$ with $H-y$. ∎

**3.5. Erdős–Sós Conjecture.** In 1959 Erdős and Gallai [29] proved that every graph $G$ having size $e(G) > \frac{1}{2}n(k-1)$ contains a path of length $k$. Motivated by this result, Erdős and Sós made the following conjecture in 1963:

CONJECTURE 3.24. *If $G$ is a graph and $e(G) > \frac{1}{2}n(k-1)$, then $G$ contains every tree $T$ of size $k$.*

Notice that this conjecture, if true, would be the best possible in the sense that for all $k$ there exists an $n$ and a graph $G$ of order $n$ such that $e(G) \leq \frac{1}{2}n(k-1)$ and $G$ contains no tree of size $k$. It suffices to consider $n = pk$ for some $p$ and $G = pK_k$, i.e. $p$ copies of $K_k$.

We shall formulate the above conjecture as a packing problem. Note first that if $e(G) > \frac{1}{2}n(k-1)$, then the size of the complementary graph satisfies $e(\overline{G}) < \frac{1}{2}n(n-k)$. Thus the Erdős–Sós Conjecture can be restated as follows.

ERDŐS–SÓS CONJECTURE. *Suppose that $G$ is a graph of order $n$ and $T$ is any tree of size $k$. If $e(G) < \frac{1}{2}n(n-k)$, then $T$ and $G$ are packable* (*into the complete graph $K_n$*).

This conjecture is still open. However, some partial results have been obtained either for some special families of trees or for particular values of the parameters $k$ and $n$, or else for some special families of graphs. Some of these results will be presented in this section.

**3.5.1.** *Special families of trees.* The Erdős–Sós Conjecture has been proved for many particular families of trees. Observe first that it is true for stars. This is an immediate consequence of the fact that if $e(G) < n(n-k)$, then the minimum degree of $G$ is at most $n-k-1$. So the star with $k$ edges can be packed with $G$ by identifying its centre with the vertex of minimum degree of $G$.

The conjecture is also true for paths. This is the above-mentioned result of Erdős and Gallai. We restate and prove it here using the packing formulation (see also [5]).

THEOREM 3.25. *If* $e(G) < \frac{1}{2}n(n-k)$, *then there exists a packing of* $G$ *and* $P_{k+1}$, *i.e. the path of size* (*length*) $k$.

Proof. Let $n$ be the smallest integer such that the claim of the theorem does not hold. Since for $k = 1$ and $k = 2$ the paths of length $k$ are stars, we may choose the smallest possible $k$ such that there is no packing of $G$ and $P_{k+1}$. By the choice of $n$ and $k$ there is a packing of $G$ and $P_k$. Given such a packing, we say that the edges of $G$ are *red* and the remaining edges are *green*.

First, we shall show that there is no green cycle of length $k$. Suppose, contrary to our claim, that such a cycle exists. Denote by $x_1, \ldots, x_k$ its vertices and let $V_1 = V(G) \setminus \{x_1, \ldots, x_k\}$. Since we cannot have a green path of length $k$, all $n-k$ edges sent to $V_1$ by each vertex on $C_k$ are red. Counting the number of edges of the graph $G' = G \setminus \{x_1, \ldots, x_k\}$ we get

$$e(G') < \frac{1}{2}n(n-k) - k(n-k) < \frac{1}{2}(n-k)(n-2k).$$

But then, by hypothesis, $G'$ would contain a path of length $k$, a contradiction.

Denote now by $x_1, \ldots, x_k$ the vertices of $P_k$ and let $V_1 = V(G) \setminus \{x_1, \ldots, x_k\}$. Since we cannot extend the path $P_k$, all edges joining $x_1$ or $x_k$ with the vertices in $V_1$ are red. The fact that there is no green cycle of length $k$ implies, in particular, that $|N_G(x_1, P_k)| + |N_G(x_2, P_k)| \geq k-1$. Suppose that $|N_G(x_1, P_k)| \geq |N_G(x_2, P_k)|$. Since $x_1$ sends $n-k$ red edges to $V_1$ we have

$$d_G(x_1) \geq n - k + \frac{k-1}{2} = \frac{n-2k-1}{2}.$$

Then $e(G - x_1) < \frac{1}{2}(n-1)(n-1-k)$ and, by the definition of $n$ and $k$, $E(G - x_1)$ contains $P_{k+1}$, a contradiction. ■

Using another theorem of Erdős and Gallai and a similar argument, Borowiecki and Vaderlind proved that Erdős–Sós Conjecture holds for all caterpillars with one leg. A proof is given in [10].

The following theorem was proved by Sidorenko in [61]. The proof presented below is the translation of the original one, given in the subgraph language, into the packing language.

THEOREM 3.26. *Suppose that* $G$ *is a graph of order* $n$ *and size* $e(G) < \frac{1}{2}n(n-k)$ *and suppose that* $T$ *is any tree of size* $k$, $k \leq n-1$, *having a vertex adjacent to* $t$ *leaves*, *where* $t \geq \frac{1}{2}(k-1)$. *Then* $T$ *and* $G$ *are packable into the complete graph* $K_n$.

Proof. Suppose that the theorem does not hold and let $n$ be the smallest integer such that for some $k$ there exist a graph $G$ and a tree $T$ of size $k$ satisfying the hypotheses of the theorem and such that there is no packing of $T$ and $G$.

Since for small $k$ the claim of the theorem is evidently true (because then $T$ is a star), we may choose $k$ as small as possible.

Finally, for fixed $n$ and $k$ we choose the largest $t$ such that there is no packing of $T$ and $G$. This is possible because if, for example, $t = k$, then $T$ is a star and a packing exists.

Denote by $x$ the vertex of $T$ adjacent to $t$ leaves of $T$, $x_1, \ldots, x_t$ say. Since $T$ is not a star, there is an end-vertex $x_{t+1}$ of $T$ not adjacent to $x$. Let $y$ be a vertex adjacent to $x_{t+1}$. Note that $x \neq y$.

Consider now the tree $T'$ obtained from $T$ by deleting the edge $yx_{t+1}$ and adding the edge $xx_{t+1}$. By the choice of $T$, there is a packing of $T'$ and $G$ into the complete graph $K_n$. The edges of $T'$ will now be considered as green edges, while the edges of $G$ will be considered as red.

We shall show that the red-degree of the vertex $y$ is at least $\frac{1}{2}(2n-k+1)$.

First observe that the $t+1$ edges $yx_i$, $i = 1, \ldots, t+1$, are red. Because, if for some $i$ the edge $yx_i$ is not red, we could replace the edge $xx_i$ by $yx_i$ in the tree $T'$ and get a packing of $T$ and $G$.

Next, it is easy to see that the $n-k+1$ vertices of $K_n$ that do not belong to $V(T)$ are connected with the vertex $y$ by a red edge, for otherwise we could add a new green edge to $E(T')$ and we would get in this way a packing of $G$ with a tree which contains $T$.

So, using the fact that $t \geq \frac{1}{2}(k-1)$, we have

$$d_G(y) \geq t+1+n-k \geq \tfrac{1}{2}(2n-k+1).$$

We now remove the vertex $y$ from the graph $G$ obtaining the graph $G'$. By hypothesis and from the above considerations it follows that

$$e(G') < \tfrac{1}{2}n(n-k) - \tfrac{1}{2}(2n-k+1) = \tfrac{1}{2}(n-1)(n-1-k).$$

Thus, the hypotheses of the Erdős–Sós Conjecture are satisfied with respect to $T$ and $G'$. Therefore, by the choice of $n$, there is a packing of $T$ and $G'$ into $K_{n-1}$, so *a fortiori* a packing of $T$ and $G$ into $K_n$, a contradiction. ∎

COROLLARY 3.27. *Let $G$ be a graph of order $n$. If $2e(G) < n(n-k)$ then $G$ is packable with all double-stars of size $k$.*

Proof. It suffices to observe that at least one of the centres of the double-star is adjacent to $t \geq \frac{1}{2}(k-1)$ leaves. Thus the result holds by the previous theorem. ∎

The above theorem also implies that the Erdős–Sós Conjecture holds for comets, i.e. the trees obtained from a star and a path by identifying one leave of the star with one leave of the path, if the length of the path is small. The first of the three propositions given below improves this fact to the case of all comets. The other two propositions concern the trees obtained from two stars by joining their centres by a path of length 2 or 3, i.e. the cases where $T$ is an s-p-s of type $(s_1, 2, s_2)$ and $(s_1, 3, s_2)$, respectively. These three propositions were proved by Saclé [53].

PROPOSITION 3.28. *Let $n, k$ be integers with $n > k$ and let $G$ be a graph of order $n$ and the size satisfying $2e(G) < n(n-k)$. If for integers $p, q$ such that $p+q = k$, $T(p,q)$ is the tree of size $k$ obtained from a path $x_0x_1\ldots x_p$ by joining $x_0$ to $q$ additional vertices $y_1, \ldots, y_q$, then there is a packing of $T(p,q)$ and $G$ in $K_n$.*

P r o o f. The case $k = n - 1$ follows from Theorem 3.9 (see also Subsection 5.3.2.B). If $p \le 2$, then the packing exists by Theorem 3.26. Let us suppose that we have $p > 2$, $n > k+1$, and suppose that $T(p,q)$ is a counterexample with the smallest $n$. Therefore we have $2\Delta(G) \le 2n-k-2$, for otherwise by deleting a vertex $x$ of maximum degree we obtain a graph $G' = G\setminus\{x\}$ of order $n-1$, with $e(G') < n(n-k)-2n+k+1 = (n-1)(n-k-1)$, and there is a packing of $T(p,q)$ and $G'$ in $K_{n-1}$ which may be easily extended to a packing of $T(p,q)$ and $G$. Suppose also that $p$ is the minimum for a counterexample with this value of $n$. Then $T(p-1,q+1)$ and $G$ are packable together. In such a packing, we say that the edges of $G$ are *red*, those of $T(p-1,q+1)$ are *green* and the remaining are *black*. Let us denote by $V$ the set of vertices of $K_n$ and put $V' = V \setminus \{x_0,\dots,x_{p-1}\}$.

The edges from $x_{p-1}$ to $V'$ must be red, otherwise we can change the colour of a black edge incident to the end-vertex of the path in $T(p-1,q+1)$ to green, and obtain the required packing. There are no more than $q-1$ black edges from any $y_i$ to $V'$, for otherwise we can pack $G$ together with $T(p,q)$ having $y_i x_0 x_1 \dots x_{p-1}$ as path. All edges $y_i x_1$ are red, for otherwise we can pack $G$ and $T(p,q)$ having $x_0 y_i x_1 x_2 \dots x_{p-1}$ as a path. Finally, if for $1 \le j \le p-3$ the edge $x_{p-1}x_j$ is black, then all the edges $y_i x_{j+1}$ are red, since otherwise there is a packing of $G$ and $T(p,q)$ with a path $x_0 x_1 \dots x_j x_{p-1} x_{p-2} \dots x_{j+1} y_i$. These properties give the following inequality for the red-degree of any vertex $y_i$: $d_G(y_i) + d_G(x_{p-1}) \ge (n-p)+(n-1-p-(q-1)+2)+p-3 = 2n-k-1$ implying $2\Delta(G) \ge 2n-k-1$, a contradiction. ∎

PROPOSITION 3.29. *Let $n$, $k$ be integers with $n > k$ and let $G$ be a graph of order $n$ and size satisfying $2e(G) < n(n-k)$. If for integers $p$, $q$ with $p \ge q$ and $p+q+2 = k$, $U(p,q)$ is the tree of size $k$ obtained from a path $x_0x_1x_2$ by joining $x_0$ to $p$ additional vertices $y_1,\dots,y_p$ and $x_2$ to other $q$ additional vertices $z_1,\dots,z_q$, then there is a packing of $U(p,q)$ and $G$ in $K_n$.*

P r o o f. The case $k = n-1$ follows from Theorem 3.9 (see also Subsection 5.3.2.B). The same is true if $q = 1$ since in this case the tree is of type $T(p,2)$ of the previous proposition.

Let us now suppose that $n > k+1$ and $q \ge 2$, and $U(p,q)$ is a counterexample with the smallest $n$ (implying, as above, that $2\Delta(G) \le 2n-k-2$), and the smallest $q$ for this value of $n$. We can now pack $G$ together with $U(p+1,q-1)$. We use the same conventions for the colours of edges in the packing, and we denote by $V$ the set of vertices of $K_n$, and we put $V' = V \setminus \{x_i, z_j\}_{0\le i\le 2,\,1\le j\le q}$.

All the edges from $x_2$ to $V'$ are red, since otherwise we can change the colour of one edge incident to this vertex from black to green, and obtain the desired packing. So we have $d_G(x_2) \ge n-k-1+p+1 = n-k+p$, and we get a contradiction if $2n-2k-1 \le 2n-2k+2p$, i.e. if $p \ge q+1$.

So let us suppose that $p = q$ (giving $k = 2(p+1)$, $\Delta(G) \le n-p-2$, thus $\delta(\overline{G}) \ge p+1 = k/2$ for the complement $\overline{G}$ of $G$). From now on we will work in $\overline{G}$ and we shall, for simplicity, denote by $d(v)$ the degree of $v$ in this graph. Since $2e(\overline{G}) > n(2p+1)$, there is at least a vertex in $V$ for which we have $d(x) \ge 2p+2$. We consider the neighbourhood of this vertex in $\overline{G}$ and we call this set $N(x)$. We distinguish three cases.

First, suppose that some $y \in N(x)$ has a neighbour $z$ in $\overline{G}$ which is not in $N(x)$. Since

$d(z) \geq p+1$, and $z$ is not adjacent to $x$ in $\overline{G}$, we may pick up $p$ neighbours of $z$ different from $x$ and $y$ and there remain at least $p$ neighbours of $x$ different from these vertices and from $y$, thus giving a packing of $G$ together with $U(p,p)$ having $xyz$ as a path.

Second, suppose that $N(x) \cup \{x\}$ is a connected component of $\overline{G}$ and that there exists a $y \in N(x)$ with $d(y) \geq p+2$. We may pick up $p+1$ neighbours of $y$ different from $x$ and select one of these neighbours, $z$ say. Now, there remain at least $p$ neighbours of $x$ different from these vertices and from $y$, and we obtain a packing where the path of $U(p,p)$ is $xyz$.

Finally, let us suppose that $N(x) \cup \{x\}$ is a connected component of $\overline{G}$ with every vertex $y \in N(x)$ satisfying $d(y) = p+1$. Therefore, the number of edges of this connected component is $|N(x)|(p+2)/2$. Let $V_1$ be the union of all the connected components of $\overline{G}$ having this property. Then $\overline{G}' = \overline{G} \setminus V_1$ is of order $n' = n - |V_1|$ and its size satisfies $2e(\overline{G}') > n(k-1) - |V_1|(p+2) \geq n'(k-1)$. Therefore, this graph must have a vertex of degree at least $k$ and this case reduces to the previous ones. ■

PROPOSITION 3.30. *Let $n, k$ be integers with $n > k$ and let $G$ be a graph of order $n$ and satisfying $2e(G) < n(n-k)$. If for any integers $p, q$ with $p \geq q$ and $p+q+3 = k$, $W(p,q)$ is the tree of size $k$ obtained from a path $x_0x_1x_2x_3$ by joining $x_0$ to $p$ additional vertices $y_1, \ldots, y_p$ and $x_3$ to another $q$ additional vertices $z_1, \ldots, z_q$, then there is a packing of $W(p,q)$ and $G$ in $K_n$.*

Proof. The case $k = n-1$ follows from Theorem 3.9 (see also Subsection 5.3.2.B). The same holds for $q = 1$, since in this case the tree is $T(p,3)$ of Proposition 3.28.

Let us take $n > k+1$, $q \geq 2$, and suppose that $W(p,q)$ is a counterexample with the least $n$ (implying, as above, $2\Delta(G) \leq 2n-k-2$), and least $q$ for this value of $n$. We can now pack $G$ together with $W(p+1,q-1)$. We use the same conventions for the colours of edges in the packing and we denote by $V$ the set of vertices of $K_n$, and put $V' = V \setminus \{x_i, z_j\}_{0 \leq i \leq 3,\, 1 \leq j \leq q}$.

As in the previous proposition, the edges joining $x_3$ with $V'$ must be red, so we have $d_G(x_2) \geq n-k-1+p+1 = n-k+p$, which is a contradiction if $2n-2k-1 \leq 2n-2k+2p$, i.e. if $p \geq q+2$.

So let us suppose that $p = q$ or $q+1$. In this case, according to the previous proposition, we can pack a tree $U(p+1,q)$ of size $k$ together with $G$. We shall use the same notation as in the previous proof, and we will work in $\overline{G}$ with the same conventions.

For any vertex $v$ we have $d(v) \geq k/2 \geq q+2$. All the edges $y_ix_1$ are red, for otherwise we can pack $W(p,q)$ having $x_0y_ix_1x_2$ as a path, and so are the edges $y_iz_j$ since otherwise we can pack $W(p,q)$ with the path $x_0y_iz_jx_2$, $p$ other vertices $y_{i'}$ adjacent to $x_0$ and $q$ other (adding $x_1$ to the set of $z_{j'}$'s) adjacent to $x_2$. Therefore, $y_i$ must have at least $q+2-2 = q$ neighbours in $\overline{G} \setminus V'$ and the theorem is proved for $p = q$.

Now if $p = q-1$, we have $d(v) \geq p+1$. If some $y_ix_2$ is red, we can conclude as above. Thus, we assume that all $x_2y_i$ are black. In this case $x_1$ has at least $p-1 = q$ neighbours in $\overline{G}$ different from $x_0$ and $x_2$ (and also from $y_i$), and one can pack $G$ together with a tree $W(p,p-1)$ with the path $x_2y_1x_0x_1$, having $p$ vertices $y_i$, with $2 \leq i \leq p+1$, adjacent to $x_2$, and $q$ other vertices adjacent to $x_1$. ■

It is mentioned in [49] that Perles proved in 1990 a fact which implies the following

Theorem 3.31. *Let $G$ be a graph with $n$ vertices such that $e(G) < \frac{1}{2}n(n-k)$ and let $T$ be a caterpillar. Then there is a packing of $G$ and $T$.* ■

This theorem improves all the previous propositions but they are still worth mentioning because of their simplicity. Moreover, the proof of the above theorem has not yet been published.

We finish this subsection by proving the Erdős–Sós Conjecture for one more special family of trees. Denote by $S(p,q)$ the tree with $1+p+q$ vertices $u, x_1, \ldots, x_p$, $y_1, \ldots, y_q$, $z_1, \ldots, z_q$ containing $p$ edges $ux_i$, $i = 1, \ldots, p$, and $q$ edges $uy_j$ and $q$ edges $y_jz_j$, $j = 1, \ldots, q$. In other words, $S(p,q)$ is a spider of diameter less than or equal to 4.

Proposition 3.32. *Let $G$ be a graph with $n$ vertices such that $e(G) < \frac{1}{2}n(n-k)$ and let $S(p,q)$ be a spider of diameter less than or equal to* 4 *and of size $k$* (*i.e.* $p+2q=k$). *Then there is a packing of $G$ and $S(p,q)$* (*into the complete graph $K_n$*).

Proof. As in the previous proofs in this section we shall use the following terminology: if a tree $T$ is packable with a graph $G$, the edges of $T$ will be called *green*, the edges of $G$ will be called *red* and the remaining edges of $K_n$ will be called *black*. In the spider $S(p,q)$, the paths of length two coming out of the *central vertex* $u$ will be called *legs*.

Suppose that $n$ is the smallest integer such that the proposition does not hold and let $k$ be the smallest integer such that there are two nonpackable graphs $G$ and $T = S(p,q)$ with $e(G) < \frac{1}{2}n(n-k)$ and $e(T) = k$.

Let us first observe that for each vertex of $G$ we must have

$$(*) \qquad 2d_G(x) \le 2n - k - 2.$$

For, if there is a vertex $v$ such that $(*)$ is not true, then it is easy to find out that the graph $G' = G \setminus \{v\}$ of order $n-1$ has size less than $\frac{1}{2}(n-1)(n-1-k)$. In this case $T$ and $G'$ would be packable in $K_{n-1}$.

Now let $T$ be chosen with $q$ as small as possible. Observe that the claim of our proposition is true for $q = 1$, so we may assume $q \ge 2$.

Consider now the graph $T' = S(p+2, q-1)$. By the choice of $T$ there is a packing of $G$ and $T'$. Let $x_i$ be an end-vertex adjacent to the central vertex of $T'$, $i = 1, \ldots, p+1$. Observe first that for each vertex $v \in V(K_n) \setminus V(T')$ we have $x_iv \in E(G)$, for otherwise $ux_iv$ could be considered as a new leg of our spider and we would have a packing of $G$ and $S(p+1,q)$, *a fortiori* of $G$ and $S(p,q)$. By the same reasoning we see that $x_ix_j \in E(G)$.

Consider now two end-vertices $x_1$ and $x_2$ adjacent to the central vertex of $T'$, and one leg of $T'$, $uy_iz_i$ say. Observe that at least two edges between $\{x_1, x_2\}$ and $\{y_i, z_i\}$ belong to $E(G)$. For, otherwise, one of the vertices $x_1, x_2$, sends two black edges to $\{y_i, z_i\}$ and the other one sends at least one black edge to $\{y_i, z_i\}$. Then it would be easy to replace the edges $ux_1$ and $ux_2$ and the leg $uy_iz_i$ by two green legs.

So, there are at least $2(q-1)$ red edges (i.e. belonging to $E(G)$) between $\{x_1, x_2\}$ and $\bigcup_{i=1}^{q}\{y_i, z_i\}$. Denote by $m$ the number of red edges that are destroyed by removing two vertices $x_1$ and $x_2$ from $V(K_n)$.

We have $m \geq 2(n-k-1)+2p+1+2(q-1)$. So $m \geq 2n-k+p-3$. If $m \geq 2n-k-2$, then it is easy to see that the graph $G' = G \setminus \{x_1, x_2\}$ with $n-2$ vertices has less than $\frac{1}{2}(n-2)(n-2-k)$ edges. Then $T$ and $G'$ are packable into $K_{n-2}$. So we have $m < 2n-k-2$, which implies $p = 0$, $k = 2q$ and $m = 2n-k-3$.

Hence, using $(*)$, we can conclude that $d_G(x_1) = d_G(x_2) = n-q-1$. Thus each of the vertices $x_1$, $x_2$ sends exactly $q$ red edges to the set $\bigcup_{i=1}^{q}\{y_i, z_i\}$. There is also a red edge between these vertices.

Assume first that one of these vertices, $x_1$ say, sends exactly one black edge to each $\{y_i, z_i\}$. In this case we can consider $x_1$ as a new centre of the spider with $q$ legs: $q-1$ legs $x_1y_iz_i$ or $x_1z_iy_i$ and $x_1ux_2$ as the remaining leg.

So there is at least one leg $y_1z_1$ such that both edges $x_1y_1$ and $x_1z_1$ are black, and, by symmetry, there is at least one other leg $y_2z_2$ such that both edges $x_2y_2$ and $x_2z_2$ are black.

We now repeat the above reasoning with respect to the tree $T''$ obtained from $T'$ by replacing the end-edge $ux_2$ by $uy_2$ and the leg $uy_2z_2$ by the leg $ux_2z_2$. Since in $T''$ the vertex $y_2$ plays the same role as, for example, $x_2$ in the spider $T'$, we can see that its red-degree is equal to $n-q-1$. Since the edge $x_2y_2$ is black, the number of red edges which are removed by removing two vertices $x_2$ and $y_2$ is greater than the analogous number in the previous case. ∎

**3.5.2.** *Particular values of parameters*

**A.** *Small values of k.* It is easy to see that all graphs of size less than 7 are members of the families listed in the previous subsection. So, we know that the conjecture is true for $k \leq 6$.

**B.** *Large values of k.* For $k = n-1$ we have $e(G) < \frac{1}{2}n$ and then $e(T)e(G) < \binom{n}{2}$ and we can apply Theorem 3.9. The proof of this case was given independently in [88].

For $k = n-2$ we have $e(G) < n$ and we can use Theorem 3.4. Note that it is actually even stronger than the Erdős–Sós Conjecture (in this case); cf. [86].

Remark. The above-mentioned applications of packing theorems justify the use of the packing formulation of the Erdős–Sós Conjecture (cf. also [86]).

In the remainder of this section we shall give the proof of the case $k = n-3$ (cf. [81]). So, the theorem we have to prove can be formulated as follows:

THEOREM 3.33. *Let $T$ be a tree of order $n-2$ and let $G$ be a graph of order $n$ and size $e(G)$. If $2e(G) < 3n$, then $T$ and $G$ are packable.*

Proof. Let us first define the graph $F = T \cup 2K_1$ of order $n$ obtained from $T$ by adding two isolated vertices. The edges of $F$ will be called *green* and the edges of $G$ *red*. So, we have to define a bijection $\sigma : V(F) \to V(G)$ such that if the edge $ab$ is green ($a$, $b$ belonging to $V(F)$) then $\sigma(a)\sigma(b)$ is not red. A vertex $x$ which is not incident with a green edge is said to be *green-free*.

The proof is by induction on $n$. Using the results given in the previous section we may conclude that the theorem is true for small values of $n$. Fix an $n$ and suppose that the result is true for all integers less than $n$.

We may also assume, of course, that $T$ is not a member of any family considered in the previous section, but only the fact that $T$ is neither a star nor a double-star will be used.

We shall consider several cases, subcases, and subsubcases.

Case 1: The graph $G$ has an isolated vertex $x$. Let $y$ be a vertex of $G$ with $d_G(y) = \Delta(G)$. Denote by $a_1, \ldots, a_r$ a path of $T$ with the largest length, and by $u$ one of the green-free vertices of $F$. Hence, in particular, we have $d(a_2) \geq 2$.

Define now the graphs $F' = F \setminus \{a_2, u\}$ and $G' = G \setminus \{x, y\}$. Consider first the case $\Delta(G) \geq 3$. An easy computation shows that the induction step can be used with respect to the graphs $F'$ and $G'$. Thus there exists a bijection $\sigma' : V(F') \to V(G')$ which is a packing of $F'$ and $G'$. By putting $\sigma(a_2) = x$, $\sigma(u) = y$ and $\sigma(v) = \sigma'(v)$ for $v \in V(F')$ we get a packing of $F$ and $G$.

If $\Delta(G) \leq 2$ then $2e(G) \leq 2(n-1)$, which implies $e(G) \leq (n-1)$, and the packing exists, for instance, by Corollary 3.22.

Case 2: $G$ has an end-vertex $x$

Subcase 2.1: $n$ is even. We put $F' = F \setminus \{a_1\}$ and $G' = G \setminus \{x\}$, where $a_1$ is as in Case 1. The fact that $2(e(G)-1) < 3(n-1)$ enables us to conclude that there exists a packing $\sigma'$ of $F'$ and $G'$. Denote by $x'$ the vertex of $G$ adjacent to $x$. If $\sigma'(a_2) \neq x'$ then we put $\sigma(a_1) = x$ and we get what we need. If $\sigma'(a_2) = x'$, we modify $\sigma'$ a little by putting $a_2$ on $x$ and $a_1$ on $v_1$ or $v_2$, where $v_1$, $v_2$ are the images of the isolated vertices of $F$ and therefore they are green-free.

Subcase 2.2: $n$ is odd. Denote by $y$ the vertex adjacent in $G$ to the end-vertex $x$.

2.2(a) $d_G(y) \geq 3$. We put $F' = F \setminus \{a_1, u_1\}$ and $G' = G \setminus \{x, y\}$, where $u_1$ is one of the green-free vertices of $F$. The condition $d_G(y) \geq 3$ enables us to apply the induction hypothesis to $F'$ and $G'$. The packing of $F$ and $G$ is now easy to define.

2.2(b) $d_G(y) = 1$, i.e. $G$ has an isolated edge $xy$. We put $F' = F \setminus \{a_1, a_2, u_1\}$ and $G' = G \setminus \{x, y, z\}$, where $z$ is a vertex of $G$ with $d_G(z) = \Delta(G)$. It is easy to define a packing $\sigma$ of $F$ and $G$ if we have a packing $\sigma$ of $F'$ and $G'$. A trivial verification shows that $G'$ has less than $3(n-3)$ edges if $\Delta(G) \geq 4$. But if $\Delta(G) = 3$ then $2e(G) \leq 2 + 3(n-2) = 3n-4$. Then $2e(G') = 2(e(G)-4) = 2e(G) - 8 \leq 3n - 12$, which is more than we need. In the case $\Delta(G) = 2$ we proceed analogously to Case 1.

2.2(c) $d_G(y) = 2$. Denote by $z$ the vertex of $G$ adjacent to $y$. The graphs $F'$ and $G'$ are defined as in the previous case. By similar reasoning we easily get a packing of $F$ and $G$ if $d_G(z) \geq 4$ (the details are left to the reader). If $d_G(z) = 1$ or $d_G(2) = 2$ we write $F' = F \setminus \{a_1\}$ and $G' = G \setminus \{y\}$. In both cases it is easy to find a packing $\sigma'$ of $F'$ and $G'$. Let $u_1, u_2$ be two isolated vertices of $F$ and let $v_i = \sigma'(u_i)$, $i = 1, 2$. So, $v_1, v_2$ are green-free. Observe that if $\sigma'(a_2) \neq x$ and $\sigma'(a_2) \neq z$, then the packing $\sigma$ can be defined by $\sigma(a_1) = y$. Suppose now, for example, that $\sigma'(a_2) = z$. Observe that at least one of the edges $zv_1$, $zv_2$ is not red (because $d_G(z) = 2$). Since $v_1$ and $v_2$ are green-free we can put $a_1$ on one of them. The same reasoning applies to the case $\sigma'(a_2) = x$ but it is not valid if $d_G(z) = 3$. In this case denote by $z_1$, $z_2$ the remaining neighbours of $z$ in $G$.

We put $F' = F \setminus \{a_1, a_r\}$ and $G' = G \setminus \{y, z\}$. The existence of a packing $\sigma'$ of $F'$ and $G'$ is easy to establish. There is only one situation where the possibility of getting a packing of $F$ and $G$ is not evident, namely the case where, for example, $\sigma'(a_2) = z_1$ and $\sigma'(a_{r-1}) = z_2$. Then we complete the packing with a slight modification of $\sigma'$ by putting $a_1$ on $y$, $a_{r-1}$ on $z$, and $a_r$ on one of the two green-free vertices $v_1$, $v_2$. This is possible because the edge $z_1z_2$ is not green unless $T$ is a double-star, and in this case the assertion follows from Corollary 3.27.

Remark. We have not used any information other than that concerning the vertices $y$ and $z$. In particular, the degree of $x$ plays no role in this case.

Case 3: $d_G(x) \geq 2$ for all vertices of $G$. Observe first that there exists a vertex $x$ of $G$ with $d_G(x) = 2$. Denote its neighbours by $x_1$ and $x_2$. We shall consider two main subcases.

Subcase 3.1: $n$ is even. Let $F' = F \setminus \{a_1\}$. Similarly to Subcase 2.1 we need to remove only one edge of $G$ in order to apply the induction hypothesis. Thus we define the graph $G'$ as follows: $V(G') = V(G) \setminus \{x\}$ and $E(G') = (E(G) \setminus \{xx_1, xx_2\}) \cup \{x_1x_2\}$. In other words, we introduce a supplementary "false" red edge. As a consequence, if for example $\sigma'(a_2) = x_1$, no neighbour of $a_2$ is put on $x_2$. Hence, as in the previous cases, we can put $a_2$ on $x$ and $a_1$ on one of the green-free vertices.

Subcase 3.2: $n$ is odd. Suppose first that $G$ has only one vertex $x$ of degree two. Then all other vertices of $G$ are of degree three. Let $y$ be a neighbour of $x$. Now we can repeat the argument of Subcase 2.2(c) with respect to the graph $G'=G\setminus\{x,y\}$ (cf. the Remark after Subcase 2.2(c)). This shows that the two neighbours of $x$ have degree at least four. This implies that $G$ has at least two vertices of degree two. Denote them by $x$ and $y$. Similarly to Subcase 2.2(c) we put $F'=F\setminus\{a_1,a_r\}$ and $G'=G\setminus\{y,z\}$. We can apply the induction hypothesis if the number of deleted edges is three, so we can add one "false" red edge to $G'$. Using this and the fact that $a_1, \ldots, a_r$ is a path of $T$ with the maximal length, we are able to define a packing of $F$ and $G$. The details are left to the reader. ∎

**C.** *Small values of* $n$. The above cases $A$ and $B$ imply that the conjecture is true for $n \leq 10$.

**3.5.3.** *Special families of graphs.* The first result concerning the Erdős–Sós Conjecture for special families of graphs was obtained by Dobson in [26], where he considered the graphs with large girth (see the next section). A short and elegant proof of this result was given by Brandt and Dobson in [15]. They proved:

THEOREM 3.34. *Let $G$ be a graph of order $n$ and with girth at least 5. If $e(G) > \frac{1}{2}n(k-1)$, then $G$ contains all trees of size $k$.* ∎

Remark. Actually, the above result has been obtained as a direct consequence of Theorem 3.38 below.

Theorem 3.34 was slightly improved in [55]:

THEOREM 3.35. *Suppose that a graph $G$ does not contain the cycle $C_4$. If $e(G) > \frac{1}{2}n(k-1)$, then $G$ contains all trees of size $k$.* ∎

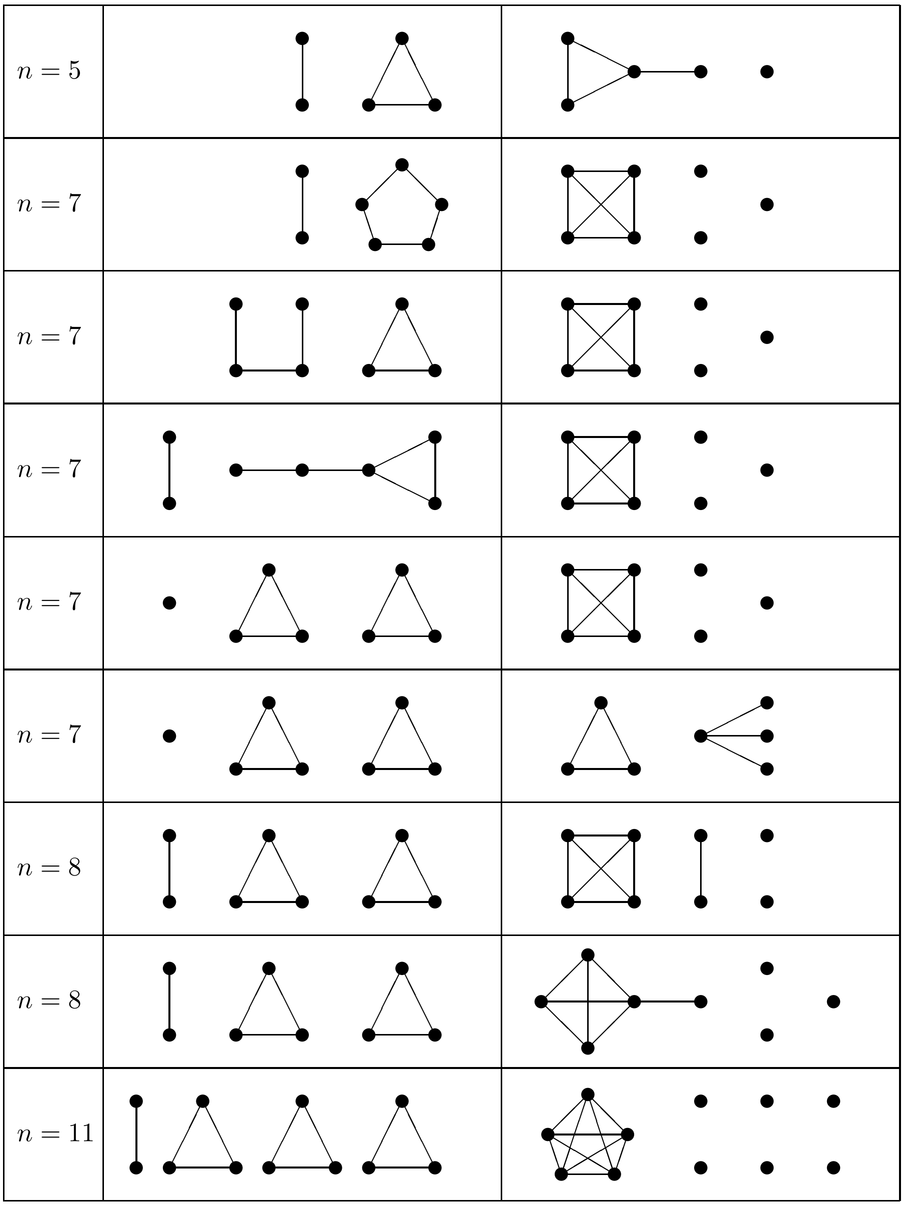



Fig. 3.4. Non-packable couples of non-isomorphic $(n, n-1)$ graphs, $n \geq 5$

**3.6. Other problems related to trees and forests.** In this section we mention without proof two problems which concern the packing of a graph with a tree or a forest. Both these problems are also closely related to the Erdős–Sós Conjecture.

**A.** *Graphs with large girth.* The following conjecture is due to Dobson [26]:

CONJECTURE 3.36. *Let $G$ be a graph with girth $g \geq 2t+1$ and let $T$ be a tree of size $k$. If $\delta(G) \geq k/t$ and $\delta(G) \geq \Delta(T)$, then $G$ contains $T$.*

The fact that the conjecture holds for $t = 1$ is well known and is often referred to as the "folklore lemma". Usually, it is presented in the following form:

THEOREM 3.37. *Let $T$ be a tree of size $k$. If $\delta(G) \geq k$, then $G$ contains $T$.* ∎

For $t = 2$ Brandt and Dobson proved [15] the following somewhat stronger result:

THEOREM 3.38. *Let $G$ be a graph with girth at least 5 and let $T$ be a tree of size $k$. If $\delta(G) \geq k/2$ and $\Delta(G) \geq \Delta(T)$, then $G$ contains $T$.* ∎

The proof of Conjecture 3.36 in the case $t = 3$ is given in [56].

For $t \geq 4$ Dobson proved in [27] that for each $\varepsilon > 0$, there exists $n$ such that every graph $G$ of order no less than $n$, with girth no less than $2t+1$ and $\delta(G) \geq k/t$, contains every tree $T$ of size $k$ with $(1-\varepsilon)k/t \geq \Delta(T)$.

**B.** *Two results on forests*. The following result was obtained by Brandt [12]. It proves another conjecture posed by Erdős–Sós. We state it in the usual subgraph form.

Set

$$f(k,n) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}.$$

THEOREM 3.39. *Let $G$ be a graph with $n$ vertices and more than $f(k,n)$ edges. Then $G$ contains every forest with $k$ edges and without isolated vertices.* ∎

Remark. Note that the bound given in the above theorem is best possible since the graphs $K_{2k-1}$ or $K_{k-1} + (n-k+1)K_1$ contain no matching of size $k$. Moreover, these graphs have been shown by Schuster [58] to be the only extremal graphs for Theorem 3.39.

We finish this section by stating without proof another result obtained by Brandt in [12]. This result generalizes the folklore lemma, i.e. Theorem 3.37.

THEOREM 3.40. *Let $F$ be a forest and $G$ be a graph, both of order $n$. If $e(F)+\Delta(G) < n$, then $F$ and $G$ are packable.* ∎

**3.7. Some generalizations.** The following theorem (proved in [67]) generalizes Theorem 3.1.

THEOREM 3.41. *Suppose that $G$ and $H$ are two $(n, n-1)$ graphs, $n \geq 5$, that are not stars. Then either $G$ and $H$ are packable or*

- *$G$ and $H$ are isomorphic to one of the four non-star graphs with $n \geq 5$ given in Figure 2.2, or else*
- *$\{G, H\}$ is one of the nine pairs shown in Figure 3.4.* ∎

The following extension of Theorem 3.4 was independently obtained by Schuster [60] and Teo and Yap [67].

THEOREM 3.42. *Suppose that $T$ is a tree of order $n$, $n \geq 5$, $T \neq S_n$, and $G$ is an $(n,n)$ graph such that $\Delta(G) < n-1$. Then either there is a packing of $T$ and $G$ or*

• *$G$ is a vertex-disjoint union of cycles $C_i$ with at least one $i$ greater than* 3 *and* $T = S'_n$,
• $n = 3k$, $G = kC_3$, *and* $T = S'_n$ *or* $T = S''_n$, *or else*
• $\{T, G\}$ *is one of the three pairs shown in Figure* 3.5. ■

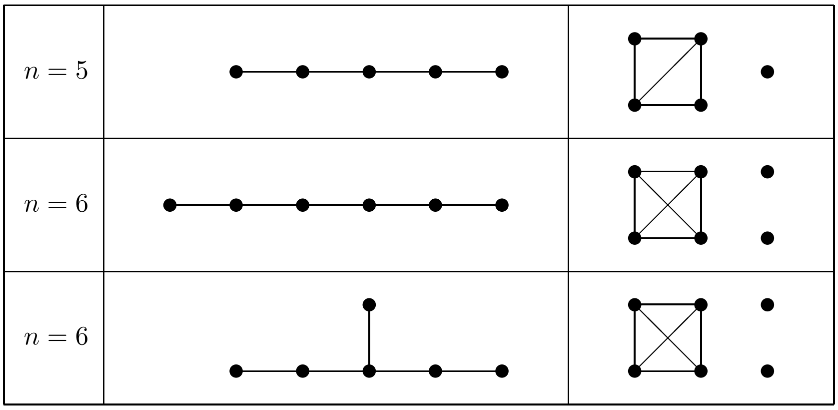



Fig. 3.5. Some forbidden pairs for Theorem 3.42

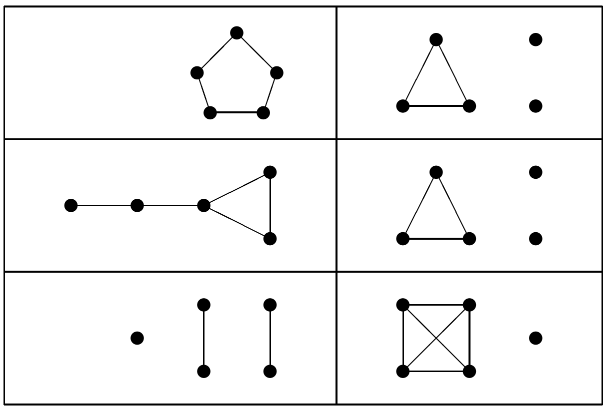


Fig. 3.6. Non-packable pairs $\{G, H\}$ of graphs of order $n$ with $e(G) \neq e(H)$, where the sum of sizes is equal to $2n - 2$. Case $n = 5$.

Finally, we shall give a complete characterization of packing of two graphs $G$ and $H$ of order $n \geq 5$ such that $\Delta(G)$, $\Delta(H) < n - 1$ and $e(G) + e(H) \leq 2n - 2$. This theorem, which extends the Main Packing Theorem, has been proved by Teo and Yap (cf. [66]).

Theorem 3.43. *Suppose that $G$ and $H$ are two graphs of order $n \geq 5$ such that $\Delta(G)$, $\Delta(H) < n - 1$ and $e(G) + e(H) \leq 2n - 2$. Then either there is a packing of $G$ and $H$ or $\{G, H\}$ is one of the following pairs of graphs*:

• $G = K_1 \cup S_{n-1}$ *and* $H = \bigcup C_i$, *where $C_i$ are cycles*,

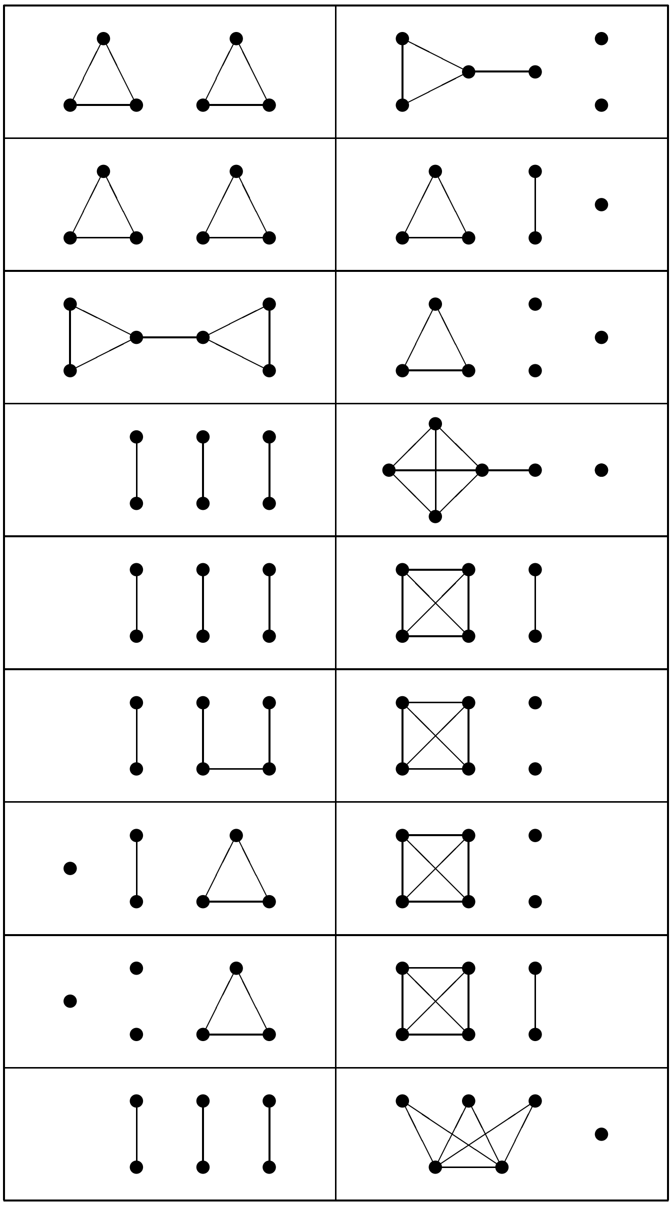


Fig. 3.7. Non-packable pairs $\{G, H\}$ of graphs of order $n$ with $e(G) \neq e(H)$, where the sum of sizes is equal to $2n-2$. Case $n = 6$.

- *$G = K_2 \cup S_{n-2}$ and $H = kC_3$, where $n = 3k$,*
- *$G$ and $H$ are isomorphic to one of the exceptional graphs of Theorem* 2.2,
- *$e(G) + e(H) = 2n - 3$ and $\{G, H\}$ is one of the exceptional pairs of Theorem* 3.20,
- *$e(G) = e(H) = n - 1$ and $\{G, H\}$ is one of the exceptional pairs of Theorem* 3.41,
- *$e(G) \neq e(H)$ and $\{G, H\}$ is one of the exceptional pairs given in Figures* 3.6–3.9. ■

We finish this section by restating the last result for $n \geq 13$.

THEOREM 3.44. *Suppose that $G$ and $H$ are two graphs of order $n \geq 5$ such that $\Delta(G)$, $\Delta(H) < n-1$ and $e(G)+e(H) \leq 2n-2$. Then either there is a packing of $G$ and $H$ or $\{G,H\}$ is one of the following three pairs of graphs*:

- $\{K_1 \cup S_{n-1}, \bigcup C_i\}$,
- $\{K_2 \cup S_{n-2}, kC_3\}$, *where* $n = 3k$,
- $\{K_3 \cup S_{n-3}, K_3 \cup S_{n-3}\}$. ■

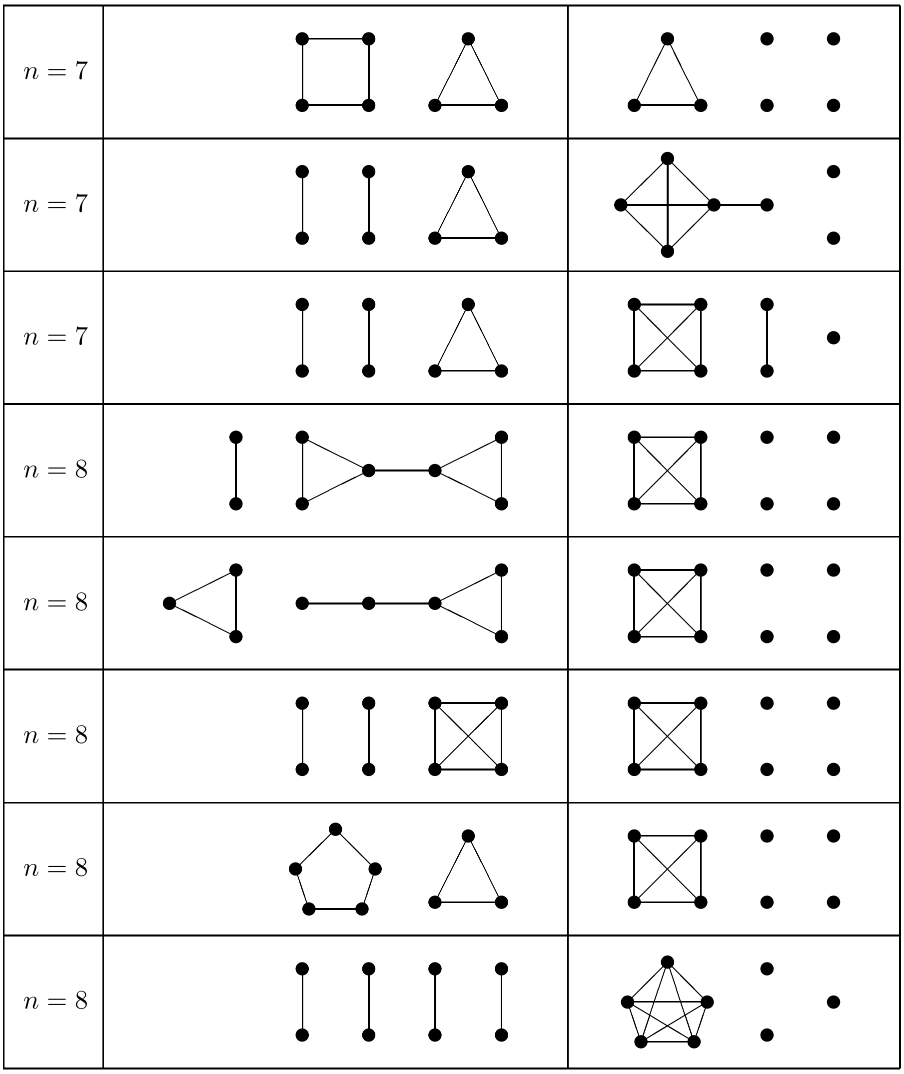



Fig. 3.8. Non-packable pairs $\{G,H\}$ of graphs of order $n$ with $e(G) \neq e(H)$, where the sum of sizes is equal to $2n-2$. Cases $n = 7$ and 8.

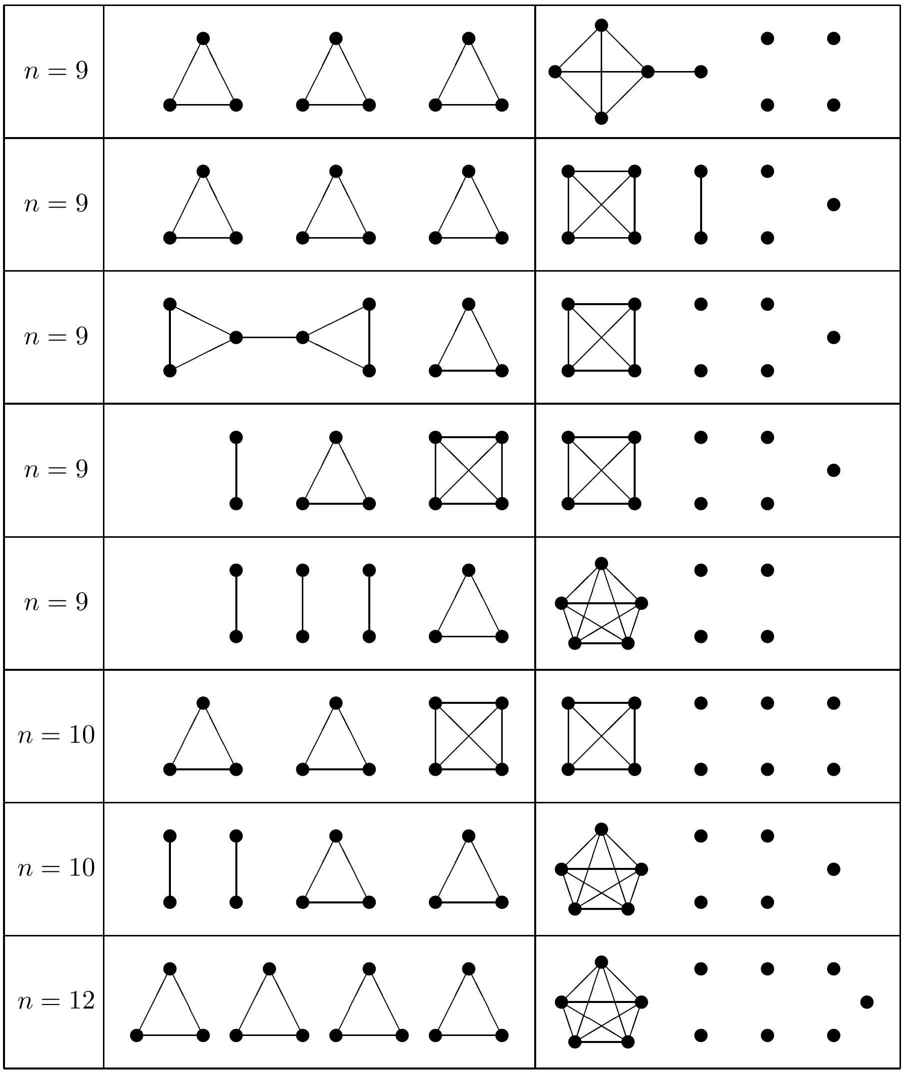



Fig. 3.9. Non-packable pairs $\{G, H\}$ of graphs of order $n$ with $e(G) \neq e(H)$ where the sum of sizes is equal to $2n-2$. Cases $n = 9, 10$ and 12.

## 4. Packing of three graphs

**4.1. Triple placement of graphs.** We begin with a theorem about 3-placement of a graph, proved in [84], which can be considered as an improvement of Theorem 2.1.

THEOREM 4.1. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n-2$ then either there exists a 3-placement of $G$ or $G$ is isomorphic to $K_3 \cup 2K_1$ or to $K_4 \cup 4K_1$.*

We shall need some additional terminology and notation. Given a 3-placement $\alpha_1, \alpha_2, \alpha_3$ of a graph $G$, we consider the graph with the vertex set $F = (a_1, \ldots, a_n)$ and the edge set $\bigcup_{i=1}^{3}\alpha_i^*(E(G))$ and we say that an edge $e$ of $F$ is *black* if $e\in\alpha_1^*(E(G))$, *red* if $e\in\alpha_2^*(E(G))$ and *green* if $e\in\alpha_3^*(E(G))$. Conversely, if there is a graph $F$ of order $n$ such that there is a partition $E_{\mathrm{b}}\cup E_{\mathrm{r}}\cup E_{\mathrm{g}}$ of the edge set of $F$ such that each of the graphs $(V(F),E_{\mathrm{b}})$, $(V(F),E_{\mathrm{r}})$, and $(V(F),E_{\mathrm{g}})$ is isomorphic to $G$, then $G$ is 3-placeable.

In every placement of a graph $G$ considered in this section, $V(G)=\{a_1,\dots,a_n\}$ and $\alpha_1$ is the identity $\mathrm{id}_{V(G)}$ of the set $V(G)$. The edges of $G$ are then said to be black in the corresponding graph $F$. So, if we have to find a 3-placement of $G$, it is sufficient to find either two permutations $\alpha_i$ of the set $V(G)$, $i$=2, 3, or the corresponding red and green graphs.

It is clear that since the star $S_n$ is an $(n,n-1)$ graph which has no 2-placement, it has no 3-placement and the theorem cannot be improved by increasing the size of $G$. On the other hand, $S_{n-1}\cup K_1$, which is an $(n,n-2)$ graph, has no 4-placement for any $n\ge 4$.

We shall need some definitions and lemmas.

Let $\mathrm{id}_{V(G)},\alpha_2,\alpha_3$ be a 3-placement of a graph $G$, and let

$$F=(V(G);E(G)\cup\alpha_2^*(E(G)))\cup\alpha_3^*(E(G)))$$

be the black-red-green graph defined by this placement. A vertex of $F$ is said to be black (red, green, respectively) if it is incident to a black (red, green, respectively) edge. It is clear that a vertex that is not black is isolated in $G$.

We say that a 3-placement $(\pi=\mathrm{id}_{V(G)},\alpha_2,\alpha_3)$ is of *type* 1 if there are three different vertices $v_1$, $v_2$ and $v_3$ such that $v_1$ is neither red nor green, $v_2$ is neither green nor black, and $v_3$ is neither black nor red.

A placement $\pi$ is said to be of *type* 2 if there are three vertices $x_1,x_2$ and $x_3$ such that $x_1$ is not black, $x_2$ is not red, and $x_3$ is not green.

A vertex of degree at least two which has a neighbour of degree one (a pendent vertex) will be called a *knot*.

LEMMA 4.2. *Every graph $G(n,n-2)$, $n\ge 3$, that is a union of a tree $T(n-1,n-2)$ and an isolated vertex $x_0$ has a 3-placement of type 2.*

Proof. We use induction on $n$. The lemma is easy to check for $n=2,3,4$. Assume that $n\ge 5$ and that the assertion holds for every integer $n'$, $2\le n'<n$. We distinguish three cases.

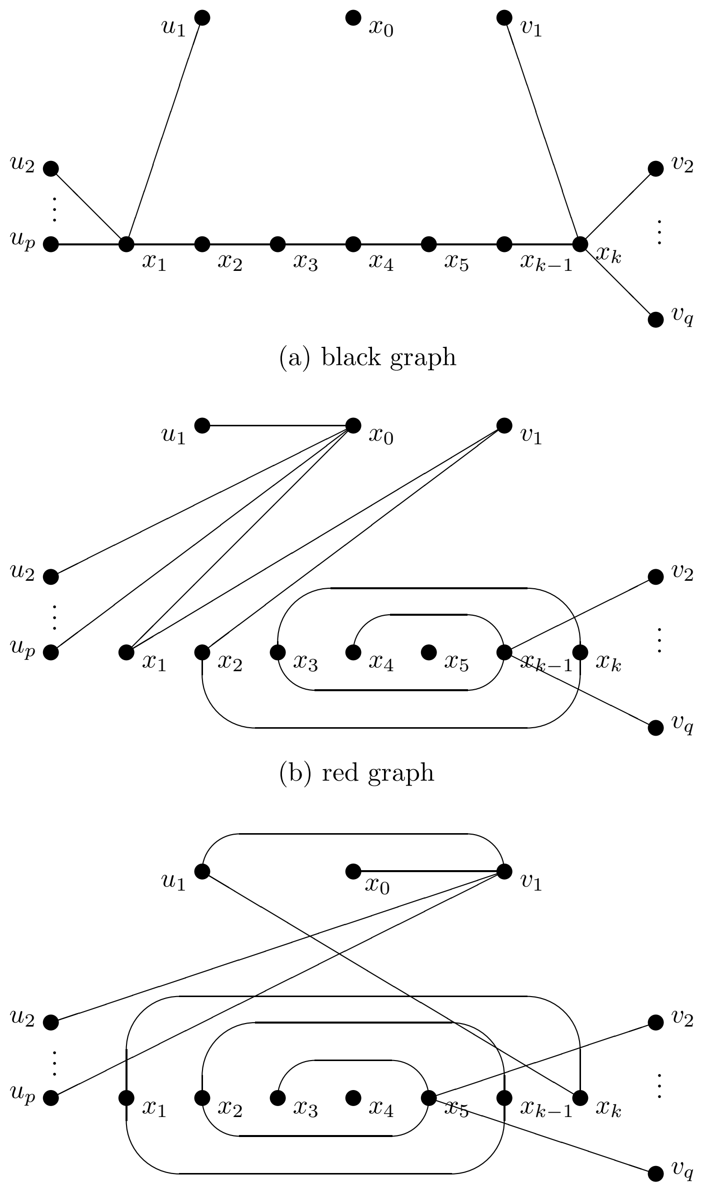
Case 1: $G=G(n,n-2)$ has at least three knots. Let $x'$, $y'$ and $z'$ be three different knots, and let $x$, $y$ and $z$ be the respective neighbours of degree one. By the induction hypothesis there is a 3-placement $(\mathrm{id}_{V(G')},\alpha_2',\alpha_3')$ of type 2 of the graph $G'=G\backslash\{x,y,z\}$. One may easily check that it is always possible to define $\alpha_i(x)$, $\alpha_i(y)$ and $\alpha_i(z)$, $i=2,3$, in such a way that $(\mathrm{id}_{V(G)},\alpha_2,\alpha_3)$, with $\alpha_i(v)=\alpha_i'(v)$ for $v\in V(G')$, is a 3-placement of $G$. Observe also that the vertices that are not black (red, green, respectively) in the 3-placement $(\mathrm{id}_{V(G')},\alpha_2',\alpha_3')$ are still not black (red, green, respectively) in the 3-placement $(\mathrm{id}_{V(G)},\alpha_2,\alpha_3)$.

Case 2: There are exactly two knots in $G$. Let $x$ and $y$ be the knots of $G$, let $u_1,u_2,\dots,u_p$ be the pendent neighbours of $x$, and let $v_1,v_2,\dots,v_q$ be the pendent neighbours of $y$, $p\ge 1$, $q\ge 1$.

Then the graph $P = T(k+2, k+1) \setminus \{u_2, \dots, u_p, v_2, \dots, v_q\}$ is a path, $P = (u_1, x = x_1, \dots, x_k = y, v_1)$, $k = n - p - q - 1 \geq 1$ (see Fig. 4.1(a)). Let us define two bijections $\alpha_2$ and $\alpha_3$ in the following way:

(1) For $x \in \{u_2, \dots, u_p, v_2, \dots, v_q\}$ we put $\alpha_2(x) = \alpha_3(x) = x$.

(2) To define $\alpha_2$ and $\alpha_3$ on the set $V(P) \cup x_0$ we distinguish three subcases.



(a) black graph

(b) red graph

(c) green graph

Fig. 4.1. Black, red and green graph

Subcase 2.1: $k = 4$, $k = 2$ or $k \geq 6$. Then put $\alpha_2(x_0) = \lfloor (k+1)/2 \rfloor + 1$, $\alpha_2(u_1) = u_1$, $\alpha_2(x_1) = x_0$ and $\alpha_2(x_2) = x_1$, $\alpha_2(x_3) = v_1$.

For $4 \leq i \leq k$ we define $\alpha_2(x_i)$ as follows: $\alpha_2(x_i) = v_{i/2}$ for $i$ even and $\alpha_2(x_i) = v_{k-(i-5)/2}$ for $i$ odd.

Finally, we put $v_1$ on $x_{k/2-1}$ for $k$ even and on $x_{(k+1)/2}$ for $k$ odd, other vertices being fixed.

The permutation $\alpha_3$ is defined as follows: $\alpha_3(x_0) = \lfloor (k+1)/2 \rfloor$, $\alpha_3(u_1) = x_0$, $\alpha_3(x_1) = v_1$, $\alpha_3(x_2) = u_1$.

For $3 \leq i \leq k$ we define $\alpha_3(x_i)$ as follows: $\alpha_3(x_i) = v_{(i-2)/2}$ for $i$ even and $\alpha_2(x_i) = v_{k-(i-3)/2}$ for $i$ odd.

Finally, we put $v_1$ on $x_{k/2+1}$ for $k$ even and on $x_{(k-1)/2}$ for $k$ odd, other vertices being fixed (see Fig. 4.1 for the case $k = 7$).

Subcase 2.2: $k = 3$. Here we put $\alpha_2(x_0) = x_3$, $\alpha_2(u_1) = u_1$, $\alpha_2(x_1) = x_0$ and $\alpha_2(x_2) = x_1$, $\alpha_2(x_3) = v_1$, $\alpha_2(v_1) = x_2$, $\alpha_3(x_0) = x_1$, $\alpha_3(u_1) = v_1$, $\alpha_3(x_1) = u_1$, $\alpha_3(x_2) = x_2$, $\alpha_3(x_3) = x_0$, $\alpha_3(v_1) = x_3$.

Subcase 2.3: $k = 5$. Put $\alpha_2(x_0) = x_3$, $\alpha_2(u_1) = u_1$, $\alpha_2(x_1) = x_0$, $\alpha_2(x_2) = x_1$, $\alpha_2(x_3) = v_1$, $\alpha_2(x_4) = x_4$, $\alpha_2(x_5) = x_2$, $\alpha_2(v_1) = x_5$, $\alpha_3(x_0) = x_2$, $\alpha_3(u_1) = v_1$, $\alpha_3(x_1) = u_1$, $\alpha_3(x_2) = x_5$, $\alpha_3(x_3) = x_0$, $\alpha_3(x_4) = x_4$, $\alpha_3(x_5) = x_1$, $\alpha_3(v_1) = x_3$.

It is clear that in each subcase $(\mathrm{id}, \alpha_2, \alpha_3)$ is a 3-placement of $G$ of type 2.

Case 3: $G$ contains exactly one knot. Then $G(n, n-2) = S_{n-1} \cup \{x_0\}$ and a 3-placement of type 2 of this graph can easily be found by the reader. ∎

LEMMA 4.3. *The following graphs are* 3*-placeable*:

(1) *the cycle* $C_7$,
(2) $K_3 \cup C_4$,
(3) *any vertex-disjoint union* $\bigcup_{i=1}^{p} K_3^i$ *of* $p \geq 3$ *triangles.*

Proof. It is easy to see that $C_7$ is 3-placeable. The 3-placement of $K_3 \cup C_4$ is represented in Figure 4.2.

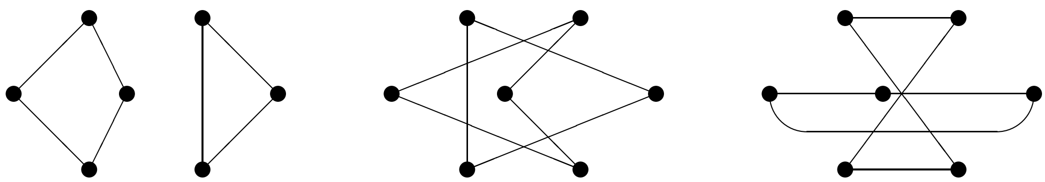


Fig. 4.2. Black, red and green copy of $K_3 \cup C_4$

Observe that the black-red-green graph is complete.

The following black-red-green graph $F$ defines a 3-placement of a vertex-disjoint union of $p \geq 3$ triangles: $V(F) = \{1, \ldots, 3p\}$, black triangles contain the vertices $3k+1, 3k+2, 3k+3$, red triangles contain the vertices $3k$, $3k+1$, $3k+5$, and finally, the vertices $3k, 3k+2, 3k+4$ determine green triangles, where $0 \leq k \leq p-1$, and the integers representing the vertices are taken modulo $3p$. ∎

In the next lemma we find the bipartite graph $H = 6K_1 * 2K_1$. Observe that $H$ is the union of three edge-disjoint cycles of length four.

LEMMA 4.4. *Let $G = C_l^1 \cup C_l^2 \cup C_l^3$ be the edge-disjoint union of three cycles of length $l$ each, $l \geq 3$. Then there are three independent edges $e_1$, $e_2$ and $e_3$ such that $e_i \in E(C_l^i)$ unless $G$ is isomorphic to $H = 6K_1 * 2K_1$.*

Proof. The proof is easy but rather lengthy, so we only give the idea. We suppose that in $G$ there is no independent edge-set $e_1, e_2, e_3$ such that $e_i \in E(C_l^i)$ for $i = 1, 2, 3$. Prove first that the set $V(C_l^1) \cap V(C_l^2) \cap V(C_l^3)$ is not empty. Then, starting from a vertex $x \in V(C_l^1) \cap V(C_l^2) \cap V(C_l^3)$ one may prove that $G$ is necessarily isomorphic to $H$. ∎

The above lemmas will help us to prove the following

LEMMA 4.5. *Every vertex-disjoint union of cycles $\bigcup_{i=1}^p C_{k_i}^i$, where $p \geq 1$, $k_i \geq 3$ for $i = 1, \ldots, p$, is 3-placeable, unless it is isomorphic either to $C_l$ with $3 \leq l \leq 6$, or to $2K_3$.*

Proof. We proceed by induction with respect to $n = \sum_{i=1}^p k_i$.

Using Lemma 4.3 we can see that the present lemma is true if $3 \leq n \leq 7$. So let us assume that $n \geq 8$, and that the lemma holds for every union of cycles of total length smaller than $n$.

If $k_i = 3$ for every $i = 1, \ldots, p$, then the lemma holds by Lemma 4.3. Thus we may assume that for at least one value of $i$, $1 \leq i \leq p$, we have $k_i \geq 4$. Suppose that $k_p \geq 4$. By the induction hypothesis, there is a 3-placement of the graph $\bigcup_{i=1}^{p-1} C_{k_i}^i \cup C_{k_p-1}^p$.

Let $G'$ be a black-red-green graph with $n-1$ vertices representing a 3-placement of $\bigcup_{i=1}^{p-1} C_{k_i}^i \cup C_{k_p-1}^p$. Then in $G'$ we have three edge-disjoint cycles $C_{k_p-1}^p(\mathrm{b})$, $C_{k_p-1}^p(\mathrm{r})$ and $C_{k_p-1}^p(\mathrm{g})$, black, red and green, respectively, corresponding to $C_{k_p-1}^p$. Suppose that we can find three independent edges $e_{\mathrm{b}}$, $e_{\mathrm{r}}$ and $e_{\mathrm{g}}$, such that $e_{\mathrm{b}} \in E(C_{k_p-1}^p(\mathrm{b}))$, $e_{\mathrm{r}} \in E(C_{k_p-1}^p(\mathrm{r}))$ and $e_{\mathrm{g}} \in E(C_{k_p-1}^p(\mathrm{g}))$. Consider a three-colour graph $G$ obtained from $G'$ by deleting the edges $e_{\mathrm{b}}$, $e_{\mathrm{r}}$ and $e_{\mathrm{g}}$, and adding a new vertex $x_0$ which we join with the black edges to the end-vertices of $e_{\mathrm{b}}$, with the red edges to the end-vertices of $e_{\mathrm{r}}$, and with the green edges to the end-vertices of $e_{\mathrm{g}}$. Then $G$ is a 3-placement of $\bigcup_{i=1}^p C_{k_i}^i$.

It remains to prove that the choice of three independent edges in the way described above is possible. We shall apply Lemma 4.4. Observe that in every edge-disjoint union $C_3^1 \cup C_3^2 \cup C_3^3$ of triangles we may always choose three independent edges $e_1$, $e_2$ and $e_3$ in such a way that $e_i \in E(C_3^i)$ and the graph obtained by deleting $e_1, e_2$ and $e_3$, and adding a new vertex $x_0$ joined with the end-vertices of $e_1, e_2$ and $e_3$, is different from the graph $H$. Also, the 3-placement of the graph $K_3 \cup C_4$ represented in Figure 4.2 does not contain $H$. Therefore we may assume that three independent edges in a black, red and green cycle of length $k_p - 1$ of the graph $G'$ always exist. ∎

LEMMA 4.6. *Let $n \geq 3$ be an integer. If every graph $G(n-1, n-3)$ is either 3-placeable or $n = 6$ and $G(n-1, n-3) = G(5,3) = K_3 \cup 2K_1$, or else $n = 9$ and $G(8,6) = K_4 \cup 4K_1$, then every graph $G(n,n) \cup 2K_1$ has a 3-placement of type 1, unless $G(n,n)$ is one of the graphs $C_n$ with $n \in \{3,4,5,6\}$ or $n = 6$ and $G(6,6) = 2K_3$ or $G(6,6) = K_4 \cup 2K_1$.*

Proof. We distinguish two cases.

Case 1: There is a vertex $z$ in $G = G(n,n)$ such that its degree is at least 3. Then the graph $G-z$ has $n-1$ vertices and at most $n-3$ edges. If there is a 3-placement of $G-z$ then a 3-placement of $G \cup 2K_2$ of type 1 is very easy to construct. If not, then $d(z,G) = 3$ and either $n-1 = 5$ and $G - z = K_3 \cup 2K_1$, or $n-1 = 8$ and $G - z = K_4 \cup 4K_1$. The reader can easily check that then either there is a 3-placement of type 1 of $G(n,n) \cup 2K_1$ or $G(n,n) \cup 2K_2 = K_4 \cup 4K_1$.

Case 2: In $G(n,n)$ there is no vertex of degree at least three. Then every vertex of $G(n,n)$ has its degree equal to two and therefore $G(n,n)$ is a union of vertex-disjoint cycles. Thus, by Lemma 4.5, either $G(n,n)$ is 3-placeable, or $3 \le n \le 6$ and $G(n,n) = C_n$, or else $n = 6$ and $G(6,6) = 2K_3$. ■

Now we may prove Theorem 4.1.

Proof of Theorem 4.1. We proceed by induction. The theorem is true for $n = 2,3,4$. Suppose that $n > 4$ and the assertion is true for every graph $G(n', n'-2)$ if $n' < n$. Let $G = G(n, n-2)$. It is easy to verify that either there are at least two isolated vertices in $G$, or $G$ is the union of $K_1$ and a tree with $n-1$ vertices, or else $G$ has a component which is a tree $T(k, k-1)$ with $2 \le k \le n-2$.

We shall consider each of these three cases separately.

Case 1: There are two isolated vertices in $G$. We can easily check that each of the graphs $C_4 \cup 2K_1$, $C_5 \cup 2K_1$, $C_6 \cup 2K_1$ and $2C_3 \cup 2K_1$ is 3-placeable. By Lemma 4.6, either $G$ is 3-placeable, or $G = C_3 \cup 2K_1$, or else $G = K_4 \cup 4K_1$.

Case 2: $G = T(n-1, n-2) \cup K_1$. The theorem follows by Lemma 4.2.

Case 3: $G$ has $T(k, k-1)$ as a component, $2 \le k \le n-2$. Denote be $R(n-k, n-k-1)$ the remaining part of the graph $G$. Thus $G = T(k, k-1) \cup R(n-k, n-k-1)$, $2 \le k \le n-2$. Consider the graph $G' = T(k, k-1) \cup \{x_1\}$, where $x_1 \notin V(G)$. We apply Lemma 4.2 to find a 3-placement of $T(k, k-1) \cup \{x_1\}$ in which $x_1$ is not black, $y_1 \in V(T(k, k-1))$ is not red and $z_1 \in V(T(k, k-1) - y_1)$ is not green.

Now consider the graph $G'' = R(n-k, n-k-1) \cup \{y_2, z_2\}$, where $\{y_2, z_2\} \cap (V(G(n,n)) \cup \{x_1\}) = \emptyset$.

Observe that every graph obtained from the graphs $C_l$, $3 \le l \le 6$, $2C_3$ and $K_4 \cup 2K_1$ by deleting an edge and adding two isolated vertices, has a 3-placement of type 1. Therefore, by Lemma 4.6, there is a 3-placement of $G''$ such that a vertex $x_2$ of $R(n-k, n-k-1)$ is neither red nor green, $y_2$ is neither green nor black, and $z_2$ is neither black nor red.

Now identify $x_1$ with $x_2$, $y_1$ with $y_2$, and $z_1$ with $z_2$, and glue together the 3-placement of $G'$ and the 3-placement of $G''$ to obtain a 3-placement of $G$. ■

**4.2. Permutation structure.** In this section we consider a theorem which improves Theorem 4.1 in a way analogous to the way in which Theorem 2.3 improves Theorem 2.2. Its proof can also be considered as another, independent, proof of Theorem 4.1 (see [79]).

THEOREM 4.7. *Let $G = (V, E)$ be a graph of order $n$, $G \ne K_3 \cup 2K_1$, $G \ne K_4 \cup 4K_1$. If $|E(G)| \le n-2$ then there exists a permutation $\sigma$ on $V(G)$ such that $\sigma^0$, $\sigma^1$, $\sigma^2$ define* *a 3-placement of $G$. Moreover, all cycles of $\sigma$ have length 3, except for one of length one if $n \equiv 1 \pmod 3$ or two of length one if $n \equiv 2 \pmod 3$.*

Proof. Throughout this proof we shall use the following terminology. A permutation $\sigma$ on $V(G)$ is said to be a *good* permutation for $G$ if $\sigma^0$, $\sigma^1$, $\sigma^2$ define a 3-placement of $G$ and $\sigma$ has all its cycles of length 3, except for one of length one if $n \equiv 1 \pmod 3$ or two of length one if $n \equiv 2 \pmod 3$. So we have to prove that if $G \neq K_3 \cup 2K_1$, $G \neq K_4 \cup 4K_1$ and $|E(G)| \leq n-2$ then there exists a good permutation for $G$. The proof is by induction on $n$. Without loss of generality we may assume that $|E(G)| = n-2$ and $G$ is not one of the exceptional graphs. The claim is true for $n = 3, 4, 5$. Assume that it is true for every graph of order $n' < n$. We shall consider three main cases.

Case 1: $n \equiv 1 \pmod 3$. (a) $G$ has a vertex $x$ with $d(x) = 1$. Consider the graph $G' = G \setminus \{x\}$ and let $\sigma'$ be a good permutation for $G'$. We put $\sigma(x) = x$ and $\sigma(v) = \sigma'(v)$ for $v \in V(G')$. Since $\sigma'$ has no fixed points, $\sigma$ is a "good" permutation for $G$.

(b) $G$ has no end-vertices. Then $G$ has at least two isolated vertices $x$, $y$. Suppose that $G$ has a vertex $z$ such that $d(z) \geq 3$ and consider the graph $G' = G \setminus \{x, y, z\}$. Let $\sigma'$ be a good permutation for $G'$. We put $\sigma = \sigma'(xyz)$. It is easy to see that $\sigma$ is a good permutation for $G$. The case where $G$ is the union of edge-disjoint cycles and two isolated vertices is considered below as Case 4.

Case 2: $n \equiv 2 \pmod 3$. (a) $G$ has two nonadjacent end-vertices $x$, $y$. We define $\sigma$, a good permutation for $G$, by $\sigma(x) = x$, $\sigma(y) = y$ and $\sigma(v) = \sigma'(v)$ for $v \in V(G')$, where $\sigma'$ is a good permutation for $G'$, $G' = G \setminus \{x, y\}$.

(b) $G$ has an isolated edge $xy$. Let $z$ be a vertex of $G$ with $d(z) \geq 2$. We consider the graph $G' = G \setminus \{x, y, z\}$. If $G'$ is not exceptional then $\sigma$ is defined by $\sigma = \sigma'(xyz)$, where $\sigma'$ is a good permutation for $G'$. The remaining cases are left to the reader.

(c) $G$ has no end-vertices. Then $G$ has at least two isolated vertices $x$, $y$. If there exists a vertex $z$ of $G$ such that $d(z) \geq 3$ then we repeat the argument of Case 2(b) with $G' = G \setminus \{x, y, z\}$. If not, then $G$ is the union of edge-disjoint cycles and two isolated vertices (see Case 4 below).

Case 3: $n \equiv 0 \pmod 3$. (a) $G$ has three end-edges $xx'$, $yy'$ and $zz'$ with end-vertices $x$, $y$, $z$ such that either $x'$, $y'$, $z'$ are distinct or $x' = y' = z'$. We consider the graph $G' = G \setminus \{x, y, z\}$ and proceed as in the above cases. A good permutation for $G$ is defined either by $\sigma = \sigma'(xyz)$ or by $\sigma = \sigma'(xzy)$.

(b) $G$ has an isolated edge or two isolated vertices. This case is similar to Case 2(b) or 2(c).

(c) $G$ contains two nontrivial trees as components and (a), (b) do not hold. Then it is easy to see that these two trees have to be isomorphic to $P_3$. Denote these two paths by $abc$ and $xyz$ and consider the graph $G' = G \setminus \{a, b, x\}$. The permutation $\sigma$ is either defined by $\sigma = \sigma'(abx)$ or by $\sigma = \sigma'(axb)$.

(d) $G$ has only one nontrivial tree component $T$. So, $G$ has at least one isolated vertex $y$. Denote by $a_1, \ldots a_k$ the longest path of $T$. Note that by (b) we may assume $k \geq 3$. If $d(a_2) \geq 3$ then we consider the graph $G' = G \setminus \{a_1, a_2, y\}$ and we get $\sigma$ by induction by putting $\sigma = \sigma'(a_1 a_2 y)$. So $d(a_2) = 2$. Thus, by symmetry, $T$ is a path for $k = 3, 4$. If

$k \geq 5$ and $d(a_i) \geq 3$ for some $i$, $3 \leq i \leq k-2$, then we can apply Subcase (a). So we have $d(a_i) = 2$ for all $i$, $2 \leq i \leq k-1$, i.e. $T$ is a path.

If $k \geq 4$ then we can consider the graph $G' = G\backslash\{a_2, a_3, y\}$ and proceed as in subcase (c).

Finally, consider the case where $T = a_1a_2a_3$. Denote by $x_1, x_2$ two nonadjacent vertices of $V(G{-}T)$ such that $d(x_i) \geq 2$ and consider the graph $G' = G\backslash\{a_1, a_2, a_3, y, x_1, x_2\}$. A good permutation for $G$ is defined now by $\sigma = \sigma'(a_1a_2x_1)(a_3x_2y)$, where $\sigma'$ is a good permutation for $G'$.

Case 4: $G$ is the union of edge disjoint cycles and two isolated vertices. Denote by $x$, $y$ the isolated vertices of $G$. Suppose that at least one component of $G$ is a cycle of length $\geq 4$ and denote by $a', a, b, b'$ four consecutive vertices on this cycle. Consider the graph $G' = G\backslash\{a, b, x\}$ and put $\sigma = \sigma'(abx)$ if $\sigma'(a') \neq b'$ and $\sigma = \sigma'(axb)$ otherwise.

So it remains to consider the case where $n = 3k + 2$ and $G = 2K_1 \cup kK_3$. For $k \geq 3$ denote by $a_ib_{i+1}c_{i+1} \pmod k$ the vertices of the $i$th triangle component of $G$. Then the permutation $(x)(y)(a_1b_1c_1)\ (a_2b_2c_2)\ \ldots (a_kb_kc_k)$ is good for $G$. The case $k = 2$ is left to the reader. This completes the proof of Theorem 4.7. ■

Theorem 4.7 implies

Corollary 4.8. *Let $G = (V, E)$ be a graph of order $n$ with $|E(G)| \leq n - 2$ such that $G$ is not isomorphic to an exceptional graph of Theorem* 4.7. *Then $G$ is contained in a* 3*-s-c graph* (*for $n \equiv 0, 1 \pmod 3$*) *or $G$ is contained in a* 3*-a-s-c graph* (*for $n \equiv 2 \pmod 3$*). ■

**4.3. 3-placement of a tree.** We are now going to consider the problem of the 3-placement of a tree.

Observe first that if there is a 3-placement of $T$ in $K_n$ then we obviously have

$$3(n-1) \leq \binom{n}{2},$$

which implies $n \geq 6$.

Moreover, since the vertex $v \in V(T)$ such that $d(v) = \Delta(T)$ must be placed with two others vertices with degrees at least one, we must assume that $\Delta(T) \leq n - 3$.

However, these evidently necessary conditions are not sufficient as is shown by the example of $S_6''$. This fact was first observed by Huang and Rosa in [45].

Wang and Sauer [71] proved the following theorem.

Theorem 4.9. *Let $T$ be a tree of order $n$, $n \geq 6$, $T \neq S_n$, $T \neq S_n'$ and $T \neq S_6''$. Then there exists a* 3*-placement of $T$.* ■

Below we give a new independent proof of Theorem 4.9, which gives some information about the permutations defining the packing (in [41] the authors mention another independent proof of this theorem).

In fact, we shall prove

Theorem 4.10. *If $T \neq S_{3k}''$, $k \geq 3$, then under the hypotheses of Theorem* 4.9 *there exists a permutation $\sigma$ on $V(T)$ such that $\sigma^0, \sigma^1, \sigma^2$ define a* 3*-placement of $T$ and $\sigma$ has* *all its cycles of length* 3, *except for one of length one if* $n \equiv 1 \pmod 3$ *or two of length one if* $n \equiv 2 \pmod 3$.

*If* $T = S''_{3k}$, $k \geq 3$, *then there exists a permutation* $\sigma$ *of* $V(T)$ *such that* $\sigma^0$, $\sigma^1$, $\sigma^2$ *define a* 3-*placement of* $T$ *and* $\sigma$ *has all its cycles of length* 3, *except for three cycles of length* 1.

The proof is by combination of a construction for some special cases and induction for the general case.

Throughout this section $T$ will be a tree of order $n$, $n \geq 6$. Without loss of generality we may assume that $T$ is not one of the exceptional graphs, i.e. $T \neq S_n$, $T \neq S'_n$, $T \neq S''_6$. In this section a permutation $\sigma$ on $V(T)$ is said to be a *good* permutation for $T$ if

(1) $\sigma^0, \sigma^1, \sigma^2$ define a 3-placement of $T$,

(2) for $T \neq S''_n$ with $n \equiv 0 \pmod 3$, $\sigma$ has all its cycles of length 3, except for one of length one if $n \equiv 1 \pmod 3$ or two of length one if $n \equiv 2 \pmod 3$,

(3) for $T = S''_n$ with $n \equiv 0 \pmod 3$, $\sigma$ has all its cycles of length 3, except for three of length one.

Using this terminology we have to prove that there exists a good permutation for $T$. We begin with the following

LEMMA 4.11. *There exists a good permutation for* $T = S''_n$.

Proof. We consider three cases.

Case (a): $n=3k+1$, $k\geq 2$. Denote by $c$ the centre of $S'$, by $c', u, a$ three consecutive vertices lying on the path of length 3 with $a$ as an end-vertex, and by $a', b, b'$ three end-vertices adjacent to $c$. The remaining $3(k-2)$ vertices are denoted by $x_i, y_i, z_i$, $1 \leq i \leq k-2$. Then a good permutation $\sigma$ is defined by

$$\sigma = (u)(abc)(a'b'c')(x_1y_1z_1)\dots(x_{k-2}y_{k-2}z_{k-2}).$$

Case (b): $n=3k+2$, $k\geq 2$. Denote by $c$ the centre of $S'$, by $c'$, $u$, $a$ three consecutive vertices lying on the path of length 3 with $a$ as an end-vertex, and by $v, a', b, b'$ four end-vertices adjacent to $c$. The remaining $3(k-2)$ vertices are denoted by $x_i$, $y_i$, $z_i$, $1 \leq i \leq k-2$. Then a good permutation $\sigma$ is defined by

$$\sigma = (u)(v)(abc)(a'b'c')(x_1y_1z_1)\dots(x_{k-2}y_{k-2}z_{k-2}).$$

Case (c): $n = 3k+3$, $k \geq 2$. Denote by $c$ the centre of $S'$, by $c'$, $u$, $a$ three consecutive vertices lying on the path of length 3 with $a$ as an end-vertex, and by $v, w, a', b, b'$ five end-vertices adjacent to $c$. The remaining $3(k-2)$ vertices are denoted by $x_i$, $y_i$, $z_i$, $1 \leq i \leq k-2$. Then a good permutation $\sigma$ is defined by

$$\sigma = (u)(v)(w)(abc)(a'b'c')(x_1y_1z_1)\dots(x_{k-2}y_{k-2}z_{k-2}). \blacksquare$$

LEMMA 4.12. *Let* $T$ *be a tree of order* $n$, $n \equiv 0 \pmod 3$. *We assume the following conditions*:

(a) $T$ *has three end-vertices* $x$, $y$, $z$ *connected by an edge to three distinct vertices* $x'$, $y'$, $z'$.

(b) $T$ *has three end-vertices* $x, y, z$ *connected by an edge to one vertex of* $T$.

(c) *$T$ has two end-vertices $x$ and $y$ such that $y$ is adjacent to a vertex $z$ of degree* 2.

*If $T$ has three vertices $x, y, z$ such that one of the conditions stated above holds and there exists a good permutation for $T' = T \setminus \{x, y, z\}$, then there exists a good permutation for $T$.*

Proof. Let us denote by $\sigma'$ a good permutation for $T'$. It is easy to see that either $\sigma = (xyz)\sigma'$ or $\sigma = (xzy)\sigma'$ is a good permutation for $T$. ■

Proof of Theorem 4.10. For $n = 6$ we have only three graphs to examine: a path $cba'ab'c'$, the caterpillar obtained from a path $ba'ab'c'$ by adding one new vertex $c$ and one new edge $ac$, and the caterpillar obtained from a path $ab'ba'$ by adding two new vertices $c$ and $c'$ and two new edges $bc$ and $b'c'$. In each case we define the good permutation by putting $\sigma = (abc)(a'b'c')$.

Assume now that the claim is true for every graph of order $n' < n$, $n > 6$.

Let $T$ be a tree on $n$ vertices. By Lemma 4.11 we may assume that $T$ is not an *exceptional graph*, i.e. $T \neq S_n$, $T \neq S'_n$ and also $T \neq S''_n$. We shall consider three main cases.

Case 1: $n \equiv 1 \pmod 3$. We choose an end-vertex $u$ in such a way that $T' = T \setminus \{u\}$ is not an exceptional graph. Let $\sigma'$ be a good permutation for $T'$. We put $\sigma(u) = u$ and $\sigma(v) = \sigma'(v)$ for $v \in V(T')$. Since $\sigma'$ has no fixed points, $\sigma$ is a good permutation for $T$.

It is easy to see that such a choice of an end-vertex $u$ is not possible in the case where $n = 7$ and $T$ is the graph obtained from a path $ab'uba'$ by adding two new vertices $c$ and $c'$ and two new edges $bc$ and $b'c'$. Then a good permutation $\sigma$ is given by $\sigma = (u)(abc)(a'b'c')$.

Case 2: $n \equiv 2 \pmod 3$. Similarly to Case 1 we choose two end-vertices $u$ and $v$ in such a way that $T' = T \setminus \{u, v\}$ is not an exceptional graph. Let $\sigma'$ be a good permutation for $T'$. We put $\sigma(u) = u$, $\sigma(v) = (v)$ and $\sigma(x) = \sigma'(x)$ for $x \in V(T')$. Since $\sigma'$ has no fixed points, $\sigma$ is a good permutation for $T$.

It is easy to see that such a choice of an end-vertex $u$ is not possible in the case where $n = 8$ and $T$ is the graph obtained from a path $ab'uba'$ by adding three new vertices $c$, $w$ and $c'$ and three new edges $bc$, $bw$ and $b'c'$. Then a good permutation $\sigma$ is given by $\sigma = (u)(v)(abc)(a'b'c')$.

Case 3: $n \equiv 0 \pmod 3$. Suppose first that $T$ has three end-vertices adjacent to one vertex of $T$. Then there are only four trees such that Lemma 4.12 is not applicable. These trees are drawn in Fig. 4.3 and it is easy to see that in each case the permutation $(abc)(xyz)(uvw)$ is good for them. The remaining cases are now easy to consider and are left to the reader. ■

**4.4. Packing three trees.** In this section we consider two special problems of packing of three trees. The general case is given in the next section.

Hobbs, Bourgeois and Kasiraj [44] proved that

THEOREM 4.13. *Trees $T_1$, $T_2$, $T_3$ of orders $n_1 < n_2 < n_3 = n$, respectively, can be packed into $K_n$.* ■

The main result of this section is the following theorem, proved first in [80].

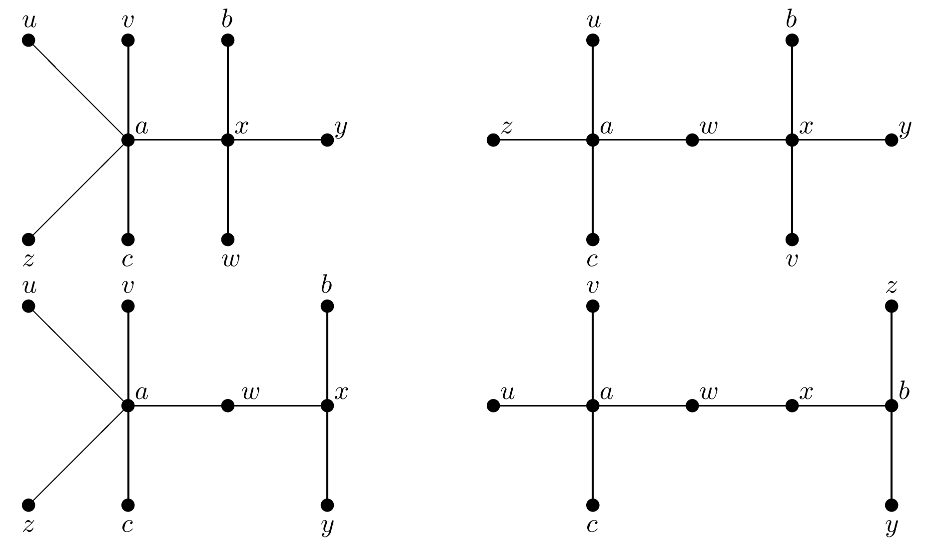



Fig. 4.3. The 3-placement is given by $(abc)(xyz)(uvw)$

THEOREM 4.14. *Let $T_1$, $T_2$, $T_3$ be three trees of order $n-1$. Then there is a packing of $T_1$, $T_2$, $T_3$ into $K_n$.*

In order to prove Theorem 4.14 we begin with a sequence of lemmas we shall need in the proof given below.

Observe first that the packing of three trees $T_1$, $T_2$, $T_3$ of size $n-2$ can be seen as a packing of three graphs $G_1$, $G_2$, $G_3$ of order $n$, where $G_i$ is obtained from $T_i$ by adding an isolated vertex, $i = 1, 2, 3$.

LEMMA 4.15. *Suppose that $G_1$, $G_2$, $G_3$ are three graphs of order $n > 5$ such that each of them has three independent end-edges. Let $G'_1$, $G'_2$, $G'_3$ be the graphs obtained from $G_1$, $G_2$, $G_3$ by removing the corresponding end-vertices. If there is a packing of $G'_1$, $G'_2$, $G'_3$ into $K_{n-3}$ then there is a packing of $G_1$, $G_2$, $G_3$ into $K_n$.*

Proof. Denote by $a_i$, $b_i$, $c_i$, $i=1,2,3$, the end-vertices of $G_1$, $G_2$ and $G_3$, respectively, corresponding to the independent end-edges and by $x_1, x_2, x_3$ three vertices of $K_n$. Let $\alpha'$, $\beta'$, $\gamma'$ be three injections defining the packing of $G'_1$, $G'_2$, $G'_3$ into $K_{n-3} = K_n \setminus \{x_1, x_2, x_3\}$.

Let $a'_i$, $b'_i$, $c'_i$, $i = 1, 2, 3$, denote the vertices adjacent to the removed end-vertices in the graphs $G_1$, $G_2$ and $G_3$, respectively. Consider the "worst" case where the sets $\alpha'(\{a'_i, a'_i, a'_i\})$, $\beta'(\{b'_i, b'_i, b'_i\})$ and $\gamma'(\{c'_i, c'_i, c'_i\})$ coincide. Let $x'$, $y'$, $z'$ be the elements of this set. Then the possibility to extend the definition of the packing follows from the fact that three matchings $a_i a'_i$, $b_i b'_i$ and $c_i c'_i$, $i = 1, 2, 3$, can be packed into the complete bipartite graph $K_{3,3}$ with colour classes $\{x, y, z\}$ and $\{x', y', z'\}$. This finishes the proof, the other cases being left to the reader. ∎

LEMMA 4.16. *Theorem 4.14 holds if one of the trees is a star.*

Proof. Without loss of generality we may assume that $T_3$ is a star. Denote by $a$ an end-vertex of $G_1$, by $b$ an isolated vertex of $G_2$, by $c$ the centre of $T_3$, and by $x$ a vertex of $K_n$. The graphs $G_1' = G_1 \setminus \{a\}$ and $G_2' = G_2 \setminus \{b\}$ can be packed into $K_{n-1} = K_n \setminus \{x\}$ by Theorem 4.13. Now it is easy to obtain a packing of $G_1$, $G_2$ and $G_3$ into $K_n$ by identifying $a$, $b$ and $c$ with $x$. ■

Lemma 4.17. *Theorem* 4.14 *holds if one of the trees is a double-star.*

Proof. Let us suppose that $T_3$ is a double-star. Denote by $a_1$, $a_2$ two end-vertices incident with two independent end-edges of $G_1$, by $b_1$ and $b_0$ an end-vertex and an isolated vertex of $G_2$, respectively, by $c_1$, $c_2$ the centres of $T_3$, and finally by $x$, $y$ two vertices of $K_n$. By Theorem 4.13, the graphs $G_1' = G_1 \setminus \{a_1, a_2\}$ and $G_2' = G_2 \setminus \{b_1, b_0\}$ can be packed into $K_{n-2} = K_n \setminus \{x, y\}$. Now it is easy to obtain a packing of $G_1, G_2$ and $G_3$ into $K_n$ by identifying first $a_1$, $a_2$ with $x$, $y$ and next $b_1$, $b_2$ and $c_1$,$c_2$ with $x$, $y$ or $y$, $x$. The details are left to the reader. ■

Lemma 4.18. *Theorem* 4.14 *holds if one of the trees is an s-p-s of type* $(s_1, 2, s_2)$ *and another one has three independent end-edges.*

Proof. Suppose that $T_3$ is an s-p-s of type $(s_1, 2, s_2)$ and $T_1$ has three independent end-edges. We proceed similarly to the proof of Lemma 4.16 but now we remove three end-vertices from $G_1$ and two end-vertices and an isolated vertex from $G_2$. ■

Lemma 4.19. *Theorem* 4.14 *holds if one of the trees is a path.*

Proof. Let $T_3$ be a path. The proof in by induction on $n$. Previous lemmas imply that Lemma 4.19 holds for small $n$. Suppose that it is true for all $n' < n$. Denote by $c_0$ the isolated vertex of $G_3$ and let $T_3$ be the path $c_1c_2 \ldots c_{n-1}$. Let $a_1, a_2$ $(b_1, b_2)$ be two end-vertices of $G_1(G_2)$ adjacent to $a_1', a_2'$ $(b_1', b_2')$, respectively. By induction hypothesis, there is a packing $\alpha'$, $\beta'$, $\gamma'$ of $G_1' = G_1 \setminus \{a_1, a_2\}$, $G_i' = G_2 \setminus \{b_1, b_2\}$ and the path $c_1c_2 \ldots c_{n-3}$ into $K_{n-2} = K_n \setminus \{x, y\}$. This is easy to place four edges from $G_1$ and $G_2$ edge-disjointly. Finally, we add to the path $\gamma'(c_1)\gamma'(c_2)\ldots\gamma'(c_{n-3})$ two edges: the first one is $\gamma'(c_{n-3})x$ or $\gamma'(c_{n-3})y$ or $\gamma'(c_1)x$ or $\gamma'(c_1)y$ and the second one is $xy$. This is not possible only in the case where the sets $\alpha'(\{a_1', a_2'\})$, $\beta'(\{b_1', b_2'\})$ and $\gamma'(\{c_1, c_{n-3}\})$ coincide. But then we can extend the path $\gamma'(c_1)\gamma'(c_2)\ldots\gamma'(c_{n-4})$ by the path $\gamma'(c_{n-4})xy\gamma'(c_0)$ of length three. ■

The following lemma was proved in [44] (as Lemma 3).

Lemma 4.20. *Suppose that* $T$ *and* $U$ *are trees of orders* $n_1 < n_2 = n$, *respectively*, $n \geq 5$, $T$ *and* $U$ *are not stars*, *and* $M$ *is a matching of size* $\leq \lfloor n/2 \rfloor$. *Then* $T$, $U$, *and* $M$ *can be packed into* $K_n$. ■

Lemma 4.21. *Theorem* 4.14 *holds if one of the trees*, $T_3$ *say*, *is a spider of diameter* $\leq 4$, *i.e. a tree with a vertex* $c$ *such that* $T_3 \setminus \{c\}$ *consists of independent edges and vertices.*

Proof. We define $G_1'$ and $G_2'$ as in Lemma 4.17 and we take $M = T_3 \setminus \{c\}$. The packing of $G_1'$, $G_2'$ and $M$ into $K_{n-2} = K_n \setminus \{x, y\}$ exists by Lemma 4.19. Now it is easy to obtain the packing we need by identifying $c$ with $x$ or $y$ and joining this vertex to one of the ends of the edges of $M$. ■

Proof of Theorem 4.14. The proof is by induction on $n$. The previous lemmas imply that Theorem 4.14 holds for small $n$. Suppose that it is true for all $n' < n$. By Lemma 4.15 we can assume that at least one of the trees $T_1$, $T_2$, $T_3$ does not contain three independent end-edges.

Case (a): Consider first the case where $T_1$ and $T_2$ have three independent end-edges and $T_3$ is a star-path-star of type $(s_1, p, s_2)$.

($a_1$) If $p \geq 4$, then we remove, as in Lemma 4.15, three end-vertices from $G_1$ and $G_2$, and we remove from $G_3$ three vertices $c_2$, $c_3$ and $c_4$, where $c_1c_2c_3 \ldots c_{p+1}$ is the path connecting the centres $c_1c_{p+1}$ of our tree. Next, we proceed as in the proof of Lemma 4.15.

($a_2$) If $T_3$ is a star, $p = 1$ or $p = 2$, we apply Lemma 4.16, 4.17 or 4.18, respectively.

($a_3$) If $p = 3$ then we can use the same argument as in Lemma 4.15 in the case where $s_1$ or $s_2$ is less than 3. For example, if $s_1 = 2$ we remove from $T_3$ the vertex $c_1$ together with two end-vertices adjacent to it, and if $s_1 = 1$ we remove from $T_3$ the vertices $c_2$ and $c_1$ with the end-vertex adjacent to $c_1$.

($a_4$) So, we may assume that $T_3$ is an s-p-s of type $(s_1, 3, s_2)$ with $s_1, s_2 \geq 3$. Denote by $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$ the end-vertices of $G_1$ and $G_2$, respectively, and by $a_0$, $b_0$ the corresponding isolated vertices. Consider now the graphs $G_1' = G_1 \setminus \{a_1, a_2, a_3, a_0\}$ and $G_2' = G_2 \setminus \{b_1, b_2, b_3, b_0\}$. If the packing of $G_1'$ and $G_2'$ (into $K_{n-4}$) exists, then it is easy to construct a packing of $G_1$, $G_2$ and $G_3$ into $K_n$. The existence of the packing of the graphs $G_1'$ and $G_2'$ into $K_{n-4}$ is guaranteed by Theorem 3.3 unless one of them, $G_1$ say, is a star. Then we try to choose the end-vertices to be removed in another way in order to avoid $G_1'$ being a star. If this is impossible then $T_1$ is a spider of diameter $\leq 4$ and we apply Lemma 4.21.

Case (b): At least two trees, $T_2$ and $T_3$ say, are star-path-stars. By Lemma 4.19 we may assume that neither $T_2$ nor $T_3$ is a path. Denote by $b_1$ a centre of $T_2$, by $c_1$ a centre of $T_3$, and by $b_{1i}$, $c_{1k}$ the end-vertices adjacent to $b_1$, $c_1$, respectively, where $i = 1, \ldots, p$, $k = 1, \ldots, q$, $p, q \geq 2$. We shall consider two subcases.

($b_1$) $T_1$ has at least two end-vertices adjacent to the same vertex $a_1$ of $T_1$. Denote them by $a_{11}, a_{12}, \ldots, a_{1r}, r \geq 2$. Consider the graphs $G_1' = G_1 \setminus \{a_1, a_{11}, a_0\}$, $G_2' = G_2 \setminus \{b_1, b_{11}, b_0\}$, $G_3' = G_3 \setminus \{c_1, c_{11}, c_0\}$, where $a_0$, $b_0$, $c_0$ denote the isolated vertices of $G_1$, $G_2$, $G_3$, respectively. Observe that the vertices $a_{11}$, $b_{11}$, $c_{11}$ are isolated in $G_1'$, $G_2'$, $G_3'$, respectively. So the packing of $G_1'$ ,$G_2'$ ,$G_3'$ (into $K_{n-3}$) exists by the induction hypothesis (notice that if the graph $G_1'$ is not a tree, then at the beginning of our procedure we can add some edges to it). Let $x$, $y$, $z$ be three vertices of $K_n$ and put $K_{n-3} = K_n \setminus \{x, y, z\}$. By identifying $a_1$, $a_{11}$, $a_0$ with $x$, $y$, $z$, and $b_1$, $b_{11}$, $b_0$ with $z$, $x$, $y$, and $c_1$, $c_{11}$, $c_0$ with $y$, $z$, $x$, respectively, we get a packing of $G_1$, $G_2$ and $G_3$ into $K_n$.

($b_2$) Let $a_1a_2 \ldots a_r$ be the longest path of $T_1$. Assume that $d(a_2) = d(a_{r-1}) = 2$ and $a_3 \neq a_{r-2}$. We proceed similarly to previous cases. The graphs $G_1'$, $G_2'$, $G_3'$ are defined as follows: $G_1' = G_1 \setminus \{a_1, a_2, a_r, a_{r-1}\}$, $G_2' = G_2 \setminus \{b_1, b_{11}, b_0, b_{21}\}$ and $G_3' = G_3 \setminus \{c_1, c_{11}, c_0, c_{21}\}$, where $b_0, c_0$ denote the isolated vertices of $G_2$, $G_3$, respectively, and $b_{21}, c_{21}$ are the end-vertices adjacent to second centre of $G_2$, $G_3$, respectively. The details are left to the reader.

Finally, if neither ($b_1$) nor ($b_2$) holds, then $T$ is a spider with $\mathrm{diam}(T) \leq 4$ and we apply once more Lemma 4.21.

This finishes the proof of Theorem 4.14. ■

**4.5. Packing three trees—general case.** In this section we present the complete result concerning packing of three trees into the complete graph $K_n$. The proof of this theorem was given by Mahéo, Saclé and the author in [48].

We shall need some additional definitions and notation in order to formulate the results. Recall that $S'_n$ is the graph obtained by subdividing one of the edges of the star $S_{n-1}$ and $S''_n$ is the tree obtained by replacing one of the edges of $S_{n-2}$ by a path of length 3. By analogy, denote by $S'''_n$ the tree obtained by replacing one of the edges of $S_{n-3}$ by a path of length 4.

Recall also that the star-path-stars are the trees obtained from a path $a_0a_1\ldots a_r$, $r \geq 1$, by adding $q \geq 1$ edges $a_ry_i$, $1 \leq i \leq q$, incident to one end-vertex of the path, and $p \geq \min\{q, 2\}$ other edges $a_0z_j$, $1 \leq j \leq p$, incident to the other end-vertex (with, obviously, $p + q + r = n - 1$). If $q \geq 2$ then we use the notation $S^r_n(p, q)$, omitting the parameters $p$ and $q$ in the cases $n = 6, 7$, where there is only one possibility (so $S^1_6 = S^1_6(2, 2)$ and $S^1_7 = S^1_7(3, 2)$). In the case when $q = 1$ this tree is a *comet* and we shall denote it by $S^{(r)}_n$ (and $S'$ for $r = 1$, $S''$ for $r = 2$ and $S'''$ for $r = 3$).

Note that the trees $S^1_n(p, q)$ are double stars. Observe also that $S'''_6$ is simply $P_6$, the path of order 6.

For $n \geq 6$ we denote by $X_n$ the tree with $n$ vertices obtained from the star $S_{n-2}$ by replacing two edges, each by a path of length 2.

By $Y_7$ we denote the tree with seven vertices obtained from the star $S_4$ by introducing three new vertices on three edges of $S_4$.

THEOREM 4.22. *If $n \geq 6$ is an integer, then one can pack any triple of trees $\mathcal{T} = (T_1, T_2, T_3)$ of order $n$ and of maximum degree at most $n - 3$ into $K_n$, except for the following* (*up to a permutation*):

- *For any $n$, the triples $(S''_n, S^1_n(a, b), T_n)$, where $T_n$ is one of the trees $S^1_n(p, q)$, $S^2_n(p, q)$, $S^3_n(p, q)$, $S''_n$, $S'''_n$.*
- *For any odd $n = 2p + 3$, the triple $(S''_n, S^2_n(p, p), S^2_n(p, p))$.*
- *For $n = 6$, the triples $(S''_6, S''_6, S''_6)$, $(P_6, X_6, S^1_6)$, $(P_6, S^1_6, S^1_6)$, $(X_6, X_6, S^1_6)$, and $(X_6, S^1_6, S^1_6)$ .*
- *For $n = 7$, the triple $(Y_7, S^1_7, S^1_7)$.* ■

**4.6. Packing three forests.** Observe that if we study the packing into the complete graph $K_n$, we can assume that all the graphs we pack are of order $n$. For, if we pack the graphs of order less than $n$, we may always add to them some isolated vertices. So, Theorems 4.13, 4.14, 4.9, 4.22 can be considered as theorems about the packing of forests.

In this section we present the general case of the packing of three forests, which generalizes all the above results concerning tree-packing. The proof, based mainly on Theorem 4.22, is given in [54].

It will be convenient to say that a triple $(F'_1, F'_2, F'_3)$ is a *subtriple* of $(F_1, F_2, F_3)$ if each $F'_i$ is a (partial) subgraph of $F_i$. We may call it a *proper subtriple* if they are not equal.

Theorem 4.23. *Let $\mathcal{F} = (F_1, F_2, F_3)$ be a triple of forests of order $n$ such that the following necessary conditions are satisfied*:

(1) $|E(F_1)| + |E(F_2)| + |E(F_3)| \le n(n-1)/2$.

(2) $\forall i = 1, 2, 3$, $\Delta(F_i) + \delta(F_{i+1}) + \delta(F_{i+2}) \le n-1$ (*where the subscripts greater than* 3 *are taken modulo* 3).

(2′) *If for any $i = 1, 2, 3$ there is the equality* (*with the same convention as in* (2)) $\Delta(F_i)+\delta(F_{i+1})+\delta(F_{i+2}) = n-1$, *then* $\delta(F_i)+\Delta(F_{i+1})+\delta'(F_{i+2})$ *and* $\delta(F_i)+\delta'(F_{i+1})+\Delta(F_{i+2})$ *are both less than or equal to* $n-1$.

(3) *If two forests are isomorphic to $K_1 \cup S_{n-1}$, the third one has a component of order less than* 3.

*Then there is a packing of these three forests into $K_n$, except, if $n \ge 6$, for the following* (*up to a permutation*):

- *for all integers $p$ and $q$ such that $2 \le p \le n-4$ and $2 \le q \le n-2$, any triple having* $(S^1_n(p, n-p-2), K_2 \cup S_{n-2}, S_q \cup S_{n-q})$ *as a subtriple*,
- *for $n$ odd, $n = 2p+3$, any triple having* $(S^2_n(p,p), S^2_n(p,p), K_2 \cup S_{n-2})$ *as a subtriple*,
- *for $n = 6, 7$ any other excluded triple of Theorem* 4.22. ■

Note that the excluded triples $(S''_n, S^1_n(a,b), T_n)$ of Theorem 4.22 belong to the first case of Theorem 4.23, and the family $(S''_n, S^2_n(p,p), S^2_n(p,p))$ to the second case.

## 5. Some special problems

**5.1. Packing a graph with its square.** In this section we give an example of a theorem where the properties of an embedding of a graph $G$ are related to $G$. This theorem, proved in [77], is also another improvement (in a new direction) of the basic Theorem 2.1.

Before stating the theorem we recall that the square $G^2$ of a graph $G = (V, E)$ is the graph with $V(G^2) = V(G)$ and $E(G^2) = \{xy : \mathrm{dist}_G(x, y) \le 2,\ x, y \in V(G)\}$.

Theorem 5.1. *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \le n-2$ then either $G$ can be embedded in the complement of its square $G^2$ or $G$ is isomorphic to one of the following exceptional graphs* (*see Figure* 5.1): $C_5 \cup 2K_1$, $C_5 \cup K_2 \cup K_1$, $C_5 \cup 2K_2$ *or* $C_5 \cup K_1 \cup K_{1,n-7}$ *for* $n \ge 11$.

If $G$ is an exceptional graph of Theorem 5.1, then it is very easy to find an embedding of $G$ in $\overline{G}$. Thus, since $G \subset G^2$, this theorem can be considered as an improvement of Theorem 2.1. In some cases this improvement is considerable. For instance if $G = K_{1,n-2} \cup K_1$ then $G^2 = K_{n-1} \cup K_1$ and in this case, after packing of $G$ and $G^2$ in $K_n$, all edges of $K_n$ except one are covered by the edges belonging either to $E(G)$ or $E(G^2)$.

Before giving the proof of Theorem 5.1 we consider some special cases formulated as lemmas.

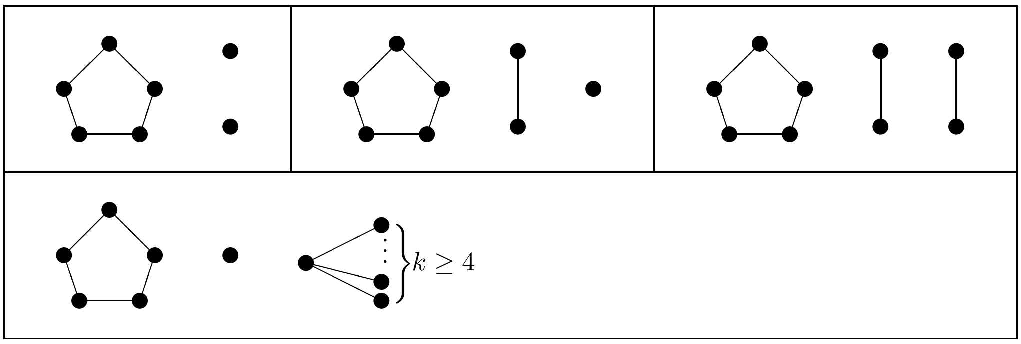



Fig. 5.1. $(n, n-2)$ graphs which are not packable with their squares

Throughout the remainder of this section we shall use the following terminology. In order to define an embedding of $G$ into the complement of its square $G^2$ it suffices to give a bijection $\alpha : V(G) \to V(G)$ such that $\alpha^*(E(G)) \cap E(G^2) = \emptyset$, where $\alpha^*(xy) = \alpha(x)\alpha(y)$. The edges of $G^2$ will be called *red* and the edges of $\alpha^*(E(G))$ will be called *black*. A vertex $x \in V(G)$ will be called *purely red* if it is incident with red edges only.

LEMMA 5.2. *For every graph $G(n, n-2)$, $n \geq 3$, that is the union of a tree $T(n-1, n-2)$ and an isolated vertex $u$, there exist two embeddings $\alpha$, $\beta$ of $G$ into the complement of $G^2$ such that if a vertex $a$ is purely red with respect to $\alpha$, then $a$ is not purely red with respect to $\beta$.*

Proof. The proof is by induction on $n$. It is easy to see that Lemma 5.2 holds in the case where $\operatorname{diam}(T) \leq 4$. Assume that $\operatorname{diam}(T) \geq 5$. Then there exist at least two vertices $a, b$ such that $G\backslash\{a\}$ and $G\backslash\{b\}$ have nontrivial components. Consider the graph $G' = G\backslash\{a, u\}$ and denote by $S_1, \ldots S_l$ the nontrivial components of $G'$. Let $N_G(a) = \{x_1, \ldots, x_l\}$. Add one new vertex $y_i$ to each component $S_i$ and apply the induction to each graph $S_i \cup \{y_i\}$. Let $z_i$ be a purely red vertex in $S_i \cup \{y_i\}$. We can assume that $z_i \neq x_i$, $i = 1, \ldots, l$. Now, by identifying the vertex $y_1$ with $z_2$, $y_2$ with $z_3, \ldots$, and $y_l$ with $z_1$ we get an embedding $\alpha'$ of $G'$ in the complement of $G'^2$ such that there is no black edge connecting any two vertices in $N_G(a)$. By putting $\alpha(a) = u$, $\alpha(u) = a$ and $\alpha(v) = \alpha'(v)$ for $v \in V(G')$ we obtain an embedding of $G$ in the complement of $G^2$ with the vertex $a$ as a purely red vertex. Applying the above argument to the graph $G\backslash\{b\}$ we get another embedding with $b$ as a purely red vertex. Then the vertex $a$ is not purely red. ■

The proofs of the following lemmas are not difficult and are left to the reader.

LEMMA 5.3. *For every graph $G(n, n-2)$, $n \geq 5$, $n \neq 7$, that is a union of a cycle $C_{n-2}$ and two isolated vertices, there is a packing of $G$ and $G^2$.* ■

LEMMA 5.4. *For every graph $G(n, n-2)$, $n \geq 8$, that is a union of at least two cycles and two isolated vertices there is a packing of $G$ and $G^2$.* ■

LEMMA 5.5. *Let $G_1$ be a union of $k$ edge-disjoint paths of length at least 2. We define a graph $G$ by adding to $G_1$ $k+1$ isolated vertices and one vertex of degree $2k$ adjacent to all ends of the $k$ paths. Then there is a packing of $G$ and $G^2$.* ■

P r o o f o f T h e o r e m 5.1. The proof is by induction on $n$. It is easy to see that an $(n, n-2)$ graph $G$ is not connected and there are at least two components of $G$ that are trees.

Suppose first that at least one component of $G$ is a nontrivial tree $T$ and let $G = T \cup R$. By Lemma 5.2 we may suppose that $R$ is not trivial. Consider the graphs $T' = T \cup \{x\}$ and $R' = R \cup \{y\}$ obtained from $T$ and $R$ by adding two new vertices $x, y$ $(x \neq y,\ x, y \notin V(G))$. Suppose that $R'$ is not an exceptional graph. By induction hypothesis, there exist an embedding $\alpha$ of $T'$ in the complement of its square and an embedding $\beta$ of $R'$ in the complement of its square. Denote by $a$ and $b$ the purely red vertices of $T'$ and $R'$ with respect to $\alpha$ and $\beta$, respectively. Identifying $x$ with $b$ and $y$ with $a$ we get a packing of $G$ and $G^2$. The case where $R'$ is an exceptional graph is left to the reader.

Suppose now that $G$ does not contain a nontrivial tree as a component. Then $G$ has at least two isolated vertices. We shall distinguish several cases.

C a s e 1: $G(n, n-2)$ contains a triangle $K_3$. Denote by $u$ one of the isolated vertices of $G$ and put $V(K_3) = \{a, b, c\}$. Observe first that we can always choose one vertex $a$ of the triangle $K_3 \in G$ in such a way that the graph $G' = G \backslash \{a, u\}$ is not exceptional.

Let $k = d_G(a)$ and let $N_G(a) = \{x_1, \ldots, x_{k-2}, b, c\}$. Consider the graph $G''$ obtained from $G'$ by adding the edges $bx_i$ (if $bx_i \notin E(G')$), $i = 1, \ldots, k-2$. The graph $G''$ has $n-2$ vertices and at most $n-4$ edges. By the induction hypothesis, there is an embedding $\alpha'$ of $G''$ in the complement of its square. Observe that all the edges connecting any two vertices of the set $N_G(a)$ are red. Since $G' \subset G''$, $\alpha'$ defines an embedding of $G'$ in the complement of $G'^2$ without black edges between the vertices belonging to $N_G(a)$. By putting $\alpha(a) = u, \alpha(u) = a$ and $\alpha(y) = \alpha'(y)$ for $y \in V(G')$ we get a packing of $G$ and $G^2$.

C a s e 2: $G(n, n-2)$ has a vertex $a$ with $d_G(a) = 3$ and does not contain a triangle $K_3$. Denote by $u, v$ two isolated vertices of $G$ and consider the graph $G' = G \backslash \{a, u, v\}$. Suppose that $G'$ is not an exceptional graph. By induction, there exists a packing $\alpha'$ of $G'$ and $G'^2$. If there are some black edges connecting vertices of $N_G(a)$, we redraw the black graph corresponding to $\alpha'$ using the isolated vertex $v$ instead of one vertex of $N_G(a)$ in such a way that there are no more black edges between vertices of $N_G(a)$. This is possible since $|N_G(a)| = 3$ and $G'$ does not contain a triangle. Now it is easy to obtain a packing of $G$ and $G^2$ by permuting the vertices $a$ and $u$. The case where $G'$ is an exceptional graph is left to the reader.

C a s e 3: $G$ does not contain a triangle $K_3$ and in $G$ there is no vertex of degree 3 or 1. Denote by $i$ the number of isolated vertices of $G$ and by $p$ the number of vertices of $G$ with degrees greater than 3. If $p = 0$, we apply Lemma 5.3 or Lemma 5.4, and if $p = 1$, we apply Lemma 5.5. So we may suppose that $p \geq 2$. Let $k = \min\{d_G(x) : d_G(x) > 3\}$. We have $2n - 4 \geq pk + 2(n - p - i)$. Hence $2i \geq 4 + p(k-2)$ and for $p \geq 2$ we get $i \geq k$.

Let $a$ be a vertex of $G$ such that $d_G(a) = k$. Denote by $u_0, u_1, \ldots, u_{k-2}$ a collection of $k-1$ isolated vertices of $G$ and consider the graph $G' = G \backslash \{u_0, u_1, \ldots, u_{k-2}, a\}$. If $G$ is not an exceptional graph then there exists a packing $\alpha'$ of $G'$ and $G'^2$. In order to obtain a packing $\alpha$ of $G$ and $G^2$ we put $\alpha(a) = u_0$, $\alpha(u_0) = a$ and use $k-2$ isolated vertices

$u_1, u_2, \ldots, u_{k-2}$ to remove the black edges (with respect to $\alpha'$) from the set $N_G(a)$, which is possible since $G'$ does not contain a triangle.

The case where $G'$ is an exceptional graph is left to the reader.

Case 4: $G$ does not contain a triangle $K_3$ and in $G$ there is no vertex of degree 3 but at least one vertex of degree 1. Denote by $G^*$ the graph obtained from $G$ in the following way: first we delete all the vertices of degree 1 in $G$, next, we delete all the vertices of degree 1 in the graph obtained and so on. Observe that $G^*$ is an $(m, m-2)$ graph which satisfies the assumptions of Case 3. Let $a$ be a vertex of $G^*$ defined as in Case 3 with $d_{G^*}(a) = k$. Denote by $u_0, u_1, \ldots, u_{k-2}$ the isolated vertices of $G^*$. Let $H$ be a tree induced (as a subgraph of $G$) by the vertex $a$ and all the vertices $x$ of $G$ such that there is a path from $x$ to $a$ in $G$ consisting of the vertices deleted in the construction of $G^*$. Let $G' = G \backslash (V(H) \cup \{u_0, u_1, \ldots, u_{k-2}\})$. If $G'$ is not an exceptional graph then there is a packing $\alpha'$ of $G'$ and $G'^2$. As in Case 3, the vertices $u_1, \ldots, u_{k-2}$ are used to remove the black edges from $N_G(a) \backslash V(H)$. It is also easy to see that a packing $\beta$ of $H' = H \cup \{u_0\}$ and $H'^2$ can be chosen in such a way that $\beta(a)$ is neither $a$ nor its neighbour. Now it is easy to define a packing of $G$ and $G^2$. ∎

**5.2. Careful packing of a graph.** Let $n$ be greater than 2. A *careful packing* of a graph $G$ is a packing of $G$, $G$, and $C_n$ into the complete graph $K_n$. In others words, it is an edge-disjoint placement of two copies of $G$ into $K_n \setminus C_n$. Geometrically speaking, if we identify the cycle $C_n$ with a convex polygon with $n$ vertices on the plane, the careful packing of $G$ gives the possibility of drawing (edge-disjointly) two copies of $G$ using only the internal edges.

Our purpose is to present the following result proved in [82].

Theorem 5.6. *Let $G$ be a graph of order $n$, $n \geq 6$. If $e(G) \leq n-2$, then there exists a careful packing of $G$ except for two graphs of order* 6: $K_3 \cup K_2 \cup K_1$ *and* $C_4 \cup 2K_1$, *and two families of graphs*: $K_{1,n-2} \cup K_1$ *and* $K_{1,n-3} \cup K_2$ (*see Fig.* 5.2).

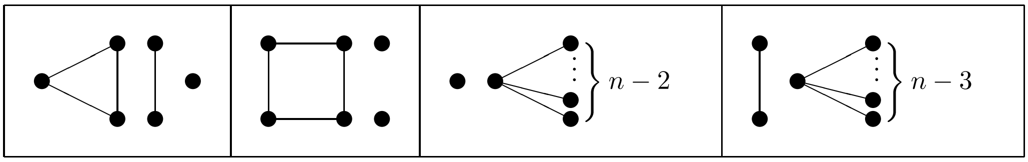



Fig. 5.2. Graphs of order $n \geq 6$ and size $e(G) \leq n-2$ not carefully packable

Before we present a proof, observe first that if we want to pack two copies of a graph $G$ together with the cycle $C_n$ then the following necessary condition must hold:

$$\Delta(G) + \delta(G) \leq n-3.$$

For, the vertex $u$ with $d(u) = \Delta(G)$ must be placed with another vertex of $G$ and with a vertex of $C_n$ of degree 2. Another obvious necessary condition is determined by the number of edges in the complete graph $K_n$. We must have $2(n-2) + n \leq \binom{n}{2}$, which implies $n \geq 6$.

So, from this point of view, there are only two small exceptional graphs in Theorem 5.6.

Theorem 5.6 implies the following

COROLLARY 5.7. *Let $G$ be a graph of order $n$, $n \geq 3$. If $e(G) \leq n-3$, then there exists a careful packing of $G$.*

P r o o f. The corollary is obvious for $n = 3$ and 4, and is easy to verify for $n = 5$. For $n \geq 6$ it follows from Theorem 5.6. ∎

Notice now that since it is very easy to find a 2-placement for exceptional graphs of Theorem 5.6, this theorem is an improvement of Theorem 2.1. On the other hand, Corollary 5.7 can also be considered as an improvement of the following theorem of Ore (cf. [9]).

THEOREM 5.8. *If $G$ is a simple graph of order $n \geq 3$ and $e(G) > \binom{n-1}{2} + 1$, then $G$ is hamiltonian.* ∎

For, note that the last theorem, restated in terms of packing, says that if $G$ is a graph of order $n$, $n \geq 3$, and $e(G) \leq n-3$, then there is a packing of $G$ into $K_n \setminus C_n$, whereas Corollary 5.7 gives a packing of *two* copies of $G$ into $K_n \setminus C_n$.

In order to prove Theorem 5.6 we begin with some simple observations formulated as lemmas. The proof of the first one is obvious.

LEMMA 5.9. *Suppose that a graph $G$ contains the cycle $C_k$ and let $u$ be a vertex not on the cycle. Denote by $|N(u, C_k)|$ the number of edges connecting $v$ with $C_k$. If $N(v, C_k) \geq \lfloor k/2 \rfloor + 1$, then the cycle $C_k$ can be extended to a cycle of length $k+1$ passing through $u$.* ∎

LEMMA 5.10. *Suppose that a graph $G$ contains the cycle $C_k$ and two vertices $u, v$ not on the cycle. If*

1. $uv \in E(G)$,
2. $|N(u, C_k)| \geq 1$, $|N(v, C_k)| \geq 1$,
3. $|N(u, C_k)| + |N(v, C_k)| \geq k + 1$,

*then the cycle $C_k$ can be extended to a cycle of length $k+2$ passing through $u$ and $v$.*

P r o o f. It is easy to see that at least one of the neighbours of the vertex $v$ on the cycle $C_k$ has as its neighbour on $C_k$ a vertex connected by an edge with the vertex $u$. The possibility of extending $C_k$ to the cycle $C_{k+2}$ is now clear. ∎

LEMMA 5.11. *If the graph $G$ has an end-vertex $x$ adjacent to the vertex $y$ of degree $d(y) \geq (n-1)/2$ and there is a careful packing of $G' = G \setminus \{x\}$, then there is a careful packing of $G$.*

P r o o f. Observe first that in the careful packing of $G'$ the image of $y$ is distinct from $y$. For, otherwise we would have too many edges adjacent to $y$ in $K_{n-1}$ (two edges of $C_{n-1}$ and at least $n-2$ edges belonging to the two copies of $G'$).

Thus it is easy to extend the packing of $G'$ (by putting $x$ on $x$) and then to extend $C_{n-1}$ by applying Lemma 5.9 to the complement of the graph $G$. ∎

Proof of Theorem 5.6. In the remainder of this section we adopt the following convention: given a careful packing of a graph $G$, we say that an edge $e$ of $K_n$ is *black* or *blue* if it belongs to the first or second copy of $G$, respectively, and that an edge $e$ of $K_n$ is *red* if it belongs to the corresponding cycle $C_n$.

The proof is by induction on $n$. Without loss of generality we may assume that all graphs in question are of maximal size $n-2$. Let us begin with the small values of $n$, i.e. $n = 6$ and $n = 7$. It is easy to see that there are five graphs of order 6 and size 4 that are not exceptional: $K_1 \cup P_5$, $K_1 \cup S_5'$, $K_2 \cup P_4$, $2P_3$, and $2K_1 \cup (S_3 + e)$. The careful packings of these graphs are shown in Figure 5.3 (the edges of $C_6$ are not marked).

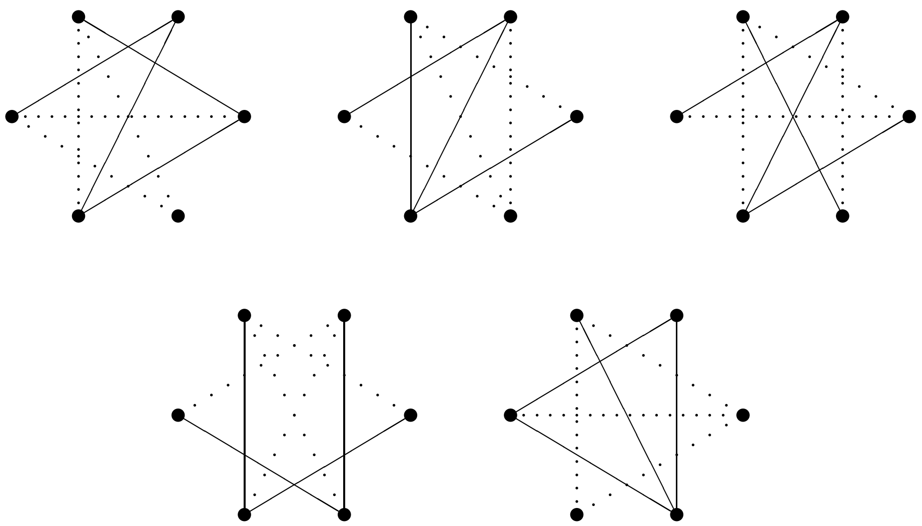


Fig. 5.3. Careful packing of graphs of order 6

Observe that they can be used to obtain careful packings of $(n, n-2)$ graphs for $n = 7$ (we can also use Lemma 5.11). The details are left to the reader.

Suppose now that the theorem is true for all $n' < n$ and let $G$ be an $(n, n-2)$ graph. Assume also that $G$ is not one of the exceptional graphs. We consider three main cases.

Case 1: $G$ has two independent end-edges. Denote the independent end-edges of $G$ by $uu'$ and $vv'$, with $u$, $v$ the corresponding end-vertices of $G$. Consider now the graph $G' = G \setminus \{u, v\}$. Suppose that there exists a careful packing $\sigma'$ for $G'$. It is easy to extend the bijection $\sigma'$ to a packing of $G$. Moreover, since the edge $uv$ is neither black nor blue, we can consider it as a red one. We also assign the red colour to $n-4$ edges connecting $u$ with $C_{n-2}$ and to $n-4$ edges connecting $v$ with $C_{n-2}$. By Lemma 5.10 (with $k = n-2$) the careful packing of $G$ exists. The case where $G'$ is an exceptional graph will be considered below as Case 3.

Case 2: $G$ does not have two independent end-edges. Since $G$ has two tree components, the above condition implies that at least one of them is trivial and the other is a star. Let $u$ be an isolated vertex of $G$ and let $x$ be the vertex defined by

$$d_G(x) = \min\{d_G(y) : y \in V(G),\ d_G(y) \geq 2\}.$$

We consider the graph $G' = G \setminus \{u, x\}$. Suppose that $G'$ is not one of the exceptional graphs. Then there exists a careful packing $\sigma'$ of $G'$. It is clear that by putting $x$ on $u$ and $u$ on $x$ we extend $\sigma'$ to a packing of $G$. We may assume that the vertices $x$ and $u$ send $n-2-d(x)$ red edges to the red cycle $C_{n-2}$ contained in $G'$. We can apply Lemma 5.10 and obtain a careful packing of $G$ if $2(n-2-d_G(x)) \geq n-1$. Hence $n-3 \geq 2d_G(x)$.

Thus we may assume that

$$(*) \qquad d_G(x) \geq \frac{n-2}{2}.$$

So, for $n \geq 7$ we have $d_G(x) \geq 3$. Consider first the case where $G$ has two trivial components.

Case 2(a): $G$ has two isolated vertices $u$, $v$. Consider first $n = 8$. The case-by-case examination shows that: either $G$ contains an end-vertex such that we can apply Lemma 5.11, or $G$ is such that the graph $G' = G \setminus \{u, x\}$ is exceptional (see Case 3). So, we may assume that $n \geq 9$. Consider now the graph $G_1 = G \setminus \{u, v, x\}$. If $G_1$ is not one of the exceptional graphs, we can use the induction hypothesis. Let $\sigma_1$ be a careful packing of $G_1$. Denote by $y_1$ a vertex of $G_1$ nonadjacent to $x$ (such a vertex exists by the definition of $x$). Without loss of generality we may assume that $y_1$ is the first vertex on the red cycle $y_1 y_2 \ldots y_{n-3}$ corresponding to the careful packing of $G_1$. Then the cycle $xy_1y_2 \ldots y_{n-3}uvx$ can be considered as a red cycle of the careful packing $\sigma$ of $G$, obtained from $\sigma'$ by putting $\sigma(x) = v$, $\sigma(v) = x$, $\sigma(u) = u$ and $\sigma(w) = \sigma'(w)$ for $w \in V(G) \setminus \{u, v, x\}$.

Case 2(b): $G$ has only one isolated vertex. Hence $G$ is of the form $K_1 \cup K_{1,r} \cup R$, where $r \geq 1$ and the graph $R$ has no isolated vertices. Moreover, since by Case 1, $R$ contains no end-vertices we may assume, by $(*)$, that either all vertices of $R$ are of degree greater than or equal to $(n-2)/2$, or $R$ is empty. In the first case, for $n > 6$, this contradicts the fact that the average degree of $R$ is equal to 2. In the second case $G$ is exceptional; a contradiction.

Case 3: $G'$ is one of the exceptional graphs, where $G'$ denotes one of the graphs defined in Cases 1 and 2 ($n \geq 8$). Without loss of generality we may assume that every other choice of two or three (for $n \geq 9$) vertices in the way described in Cases 1 and 2 also leads to one of the exceptional graphs. Of course, we can proceed as in Case 2 also in the case where the graph $G$ has two independent end-edges.

Recall that $G$ itself is not an exceptional graph.

The case-by-case examination shows that then $G$ belongs to one of the following families of graphs: $P_3 \cup K_{1,n-4}$, $K_1 \cup S'_{n-1}$, $2K_1 \cup (K_{1,n-3} + e)$, $K_1 \cup K_3 \cup K_{1,n-5}$, or $n = 8$ and $G$ is isomorphic to $4K_1 \cup K_4$, $2K_1 \cup 2K_3$ $2K_2 \cup C_4$, $K_2 \cup P_3 \cup C_3$ or $3K_1 \cup K_{2,3}$.

Observe that in all the graphs belonging to the above-mentioned families, except for $K_1 \cup K_3 \cup K_{1,3}$, there is a vertex of degree greater than or equal to $n-4$, so we can apply Lemma 5.11 (since $n \geq 8$).

The careful packings of $4K_1 \cup K_4$, $2K_2 \cup C_4$, or $2K_1 \cup 2K_3$ are very symmetric and easy to find.

The careful packing of $K_2 \cup P_3 \cup C_3$, as well as the careful packing of $K_1 \cup K_3 \cup K_{1,3}$, are shown in Fig. 5.4.

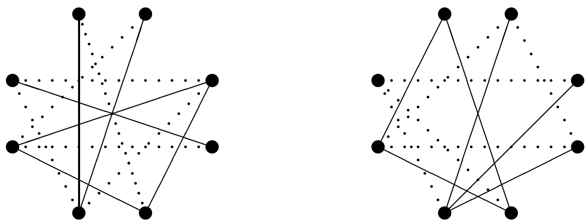


Fig. 5.4. Carefull packing of $K_2 \cup P_3 \cup C_3$ and $K_1 \cup K_{1,3} \cup C_3$

Finally, the careful packing of $3K_1 \cup K_{2,3}$ can be easily obtained from the careful packing of $2K_1 \cup K_{1,3}$ into $K_6$.

This completes the proof of Theorem 5.6. ■

### 5.3. Packing of sequences of trees

**5.3.1.** *Tree Packing Conjecture.* By the *Tree Packing Conjecture* (*TPC*) we mean the following conjecture, posed by Gyárfás in [37], which still remains open.

Conjecture 5.12. *Let $T_i$ denote a tree of order $i$. The sequence of trees $T_2, T_3, \dots, T_n$ can be packed into $K_n$.*

If we add up the sizes of the trees of the sequence $T_2, T_3, \dots, T_n$ we obtain the size of the complete graph $K_n$. So, in fact, TPC is a *decomposition* problem. However, we consider it here for two reasons. First, some typical packing problems are motivated by this conjecture. For example, the theorem about packing of three trees of different sizes (Theorem 4.13). Second, a nice proof (due to Zaks and Liu [87]) of a special case of TPC, namely where the trees are either paths or stars, gives a technique which is new comparing with those used so far.

In order to present this proof we need some additional definitions.

Let $G$ be a graph with the vertex set $V = \{v_1, \dots, v_n\}$ and the edge set $E$. Denote by $A(G) = (a_{ij})$ the adjacency matrix of $G$, such that $a_{ij} = 1$ if $v_i v_j \in E$ and $a_{ij} = 0$ otherwise. Note that $a_{ii} = 0$ for all $i$. We may use only the upper right part of $A(G)$, that is, $(a_{ij})$ for $1 \le i < j \le n$, because $G$ is undirected, and we denote it by $A^R(G)$.

In all the figures we use the notation $i$ for $v_i$. Moreover, in packing problems it will be useful to consider the generalized adjacency matrix (and its upper right part) where $a_{ij}$ is considered as a blank square (and not as a number) with the convention that (in the figure) this square is empty if the edge $v_i v_j \notin E(G)$, and if $v_i v_j \in E(G)$, then the square is marked by a symbol (the same for all edges of $G$, but not necessarily "1"). This convention allows us to represent the packing of two (or more) graphs in the same figure, that is, to make the distinction between the one's belonging to different graphs.

For example, in the left-hand part of Fig. 5.5 the edges of $P_6$ are marked by "$*$" and the edges of $S_5$ by "$+$". In the right-hand part of Fig. 5.5 we have a packing of the star $S_7$ and the path $P_6$.

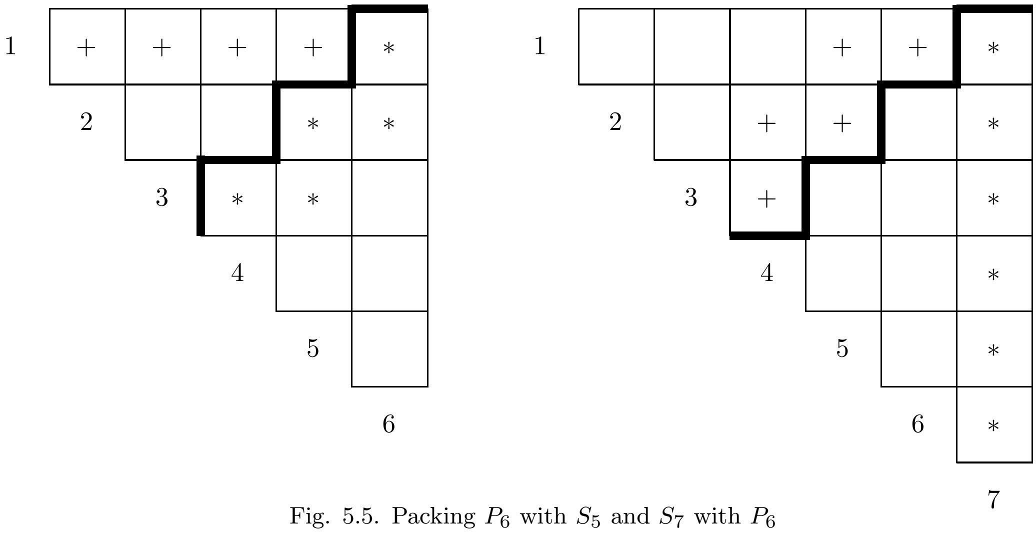



Fig. 5.5. Packing $P_6$ with $S_5$ and $S_7$ with $P_6$

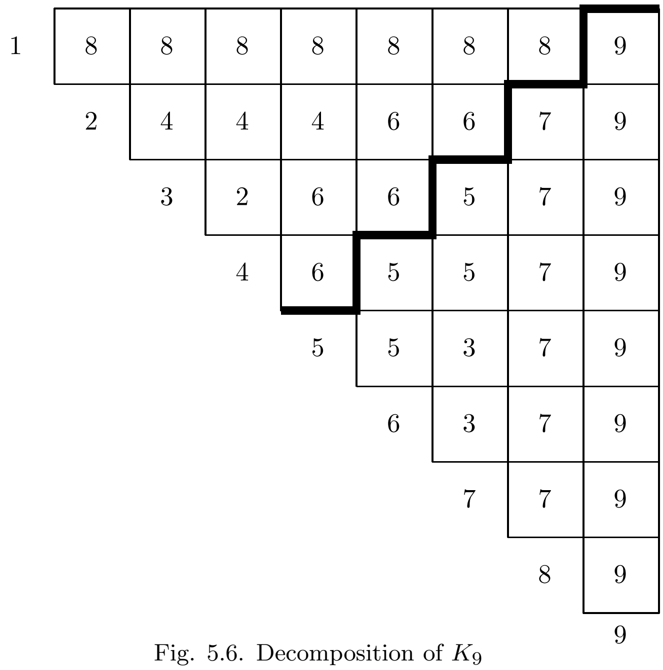



Fig. 5.6. Decomposition of $K_9$

We are now able to prove the following theorem.

THEOREM 5.13. *$K_n$ can be decomposed into $T_2, T_3, \ldots, T_n$, where $T_i$ is either $S_i$ or $P_i$ for $i = 2, \ldots, n$.*

P r o o f. We divide $A^R(K_n)$ into two blocks:

$$A_1(n)=\{a_{ij} \mid 1 \le i < j \le n,\ i+j \le n\},$$

$$A_2(n)=\{a_{ij} \mid 1 \le i < j \le n,\ i+j > n\}.$$

We begin with packing of the two biggest trees, one in $A_1(n)$ and the other in $A_2(n)$, in the way shown in Figure 5.5 for $n = 6$ and $n = 7$. We are left with two blocks of blank squares of the same shape as $A_1(n)$ and $A_2(n)$ but of the size $n - 2$. So we can pack all the rest of the trees by the induction hypothesis. ∎

Example 1. In Figure 5.6 we show the decomposition of $K_9$ into $P_9$, $P_8$, $P_7$, $S_6$, $P_5$, $S_4$, $S_3$, $S_2$ (the edges of a tree of order $i$ are marked by $i$).

Let us also mention some other partial results on TPC concerning special sequences of trees.

• Gyárfás and Lehel [37] showed that if $T_2, T_3, \ldots, T_n$ are trees and if all except two of these trees are stars, then there is a packing of these trees in $K_n$.

• Straight [64] proved TPC for $n \le 7$, and showed that if $T_2, T_3, \ldots, T_n$ are trees and each of these trees is a path, a star, or a caterpillar with maximum degree 3 in which $\lfloor \frac{1}{3}(i-3) \rfloor$ vertices have degree 3 and in which the vertices of degree two are adjacent to the end-vertices, then they can be packed in $K_n$.

• Hobbs, Bourgeois and Kasiraj proved in [44] that if $T_2, T_3, \ldots, T_n$ are trees, $T_i$ having order $i$, such that at most one of the trees has diameter more than 3, then they can be packed in $K_n$.

• Dobson verified in [28] that the Tree Packing Conjecture is true for all sequences of trees $T_2, T_3, \ldots, T_n$ such that for each $i$ there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $i - \sqrt{6(i-1)}/4$ isolated vertices.

An analogous technique to that in Theorem 5.13 was used in [87] to obtain a decomposition of the complete bipartite graph $K_{n,n}$ into a sequence of paths $P_3, \ldots, P_{2n-1}$ of even lengths.

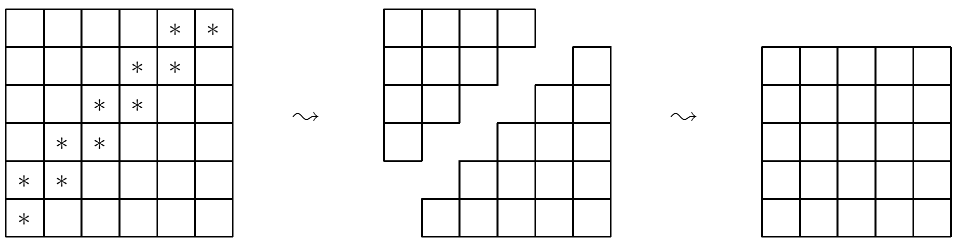


Fig. 5.7. From $K_{6,6}$ to $K_{5,5}$

Note that in $A^R(K_{p,q})$ we have a full $p \times q$ rectangular array of 1's, with 0's outside. In our discussion we work only with this rectangle (matrix) with the same convention as previously. In particular, we shall replace the one's corresponding to the edges of a graph by another symbol (for example, a star as in Fig. 5.7, or by the order of the graph as in Fig. 5.8).

We will prove the following theorem ([87]).

Theorem 5.14. *$K_{n,n}$ can be decomposed into $P_1, P_3, \ldots, P_{2n-1}$.*

| | | | | | |
|---|---|---|---|---|---|
| 4 | 6 | 8 | 10 | 12 | 12 |
| 6 | 8 | 10 | 12 | 12 | 10 |
| 8 | 10 | 12 | 12 | 10 | 8 |
| 10 | 12 | 12 | 10 | 8 | 6 |
| 12 | 12 | 10 | 8 | 6 | 4 |
| 12 | 10 | 8 | 6 | 4 | 2 |

Fig. 5.8. Decomposition of $K_{6,6}$ into $P_4, P_6, \ldots, P_{12}$

Proof. The sequence of squares marked by "$*$" in Figure 5.7 that begins in the bottom left square and ends in the upper right square corresponds to a path $P_{2n-1}$ of length $2n-1$ in $K_{n,n}$. After omitting it we are left with two separated parts of the $n \times n$ matrix (cf. Fig. 5.7). Now, we can move these parts towards each other getting an $(n-1) \times (n-1)$ matrix, and the proof is thus completed by induction. ■

Example 2. The decomposition of $K_{6,6}$ into $P_1, P_3, P_5, P_7, P_9$ is shown in Figure 5.8.

Finally, let us mention two conjectures given in [33] that together imply the TPC:

1. For $n$ even, any sequence of trees $T_1$, $T_3$, $\ldots$, $T_{n-1}$ can be packed into the half-complete graph $H_n$.
2. For $n$ odd, any sequence of trees $T_2$, $T_4$, $\ldots$, $T_{n-1}$ can be packed into the half-complete graph $H_n$.

The *half-complete graph* $H_n$ is defined as follows (cf. [33]): For each $n \geq 1$, $H_{2n}$ is the graph with $2n$ vertices and the degree sequence $(2n-1, 2n-2, \ldots, n+1, n, n-1, \ldots, 2, 1)$, and $H_{2n+1}$ is the graph with $2n+1$ vertices and the degree sequence $(2n, 2n-1, \ldots, n+1, n, n, n-1, \ldots, 2, 1)$.

It is easy to see that $H_{2n}$ and $H_{2n+1}$ are unique (up to isomorphism) and that $H_n$ and $H_{n+1}$ are complementary relative to $K_{n+1}$.

The conjectures have been proved for all trees when $n \leq 9$ (which improves the Straight's result mentioned above) and for some special types of trees (cf. [33]).

Remark. Observe that the problem of packing of a sequence of trees into half-complete graphs has already been considered in Theorem 5.13. Indeed, the two blocks defined in the proof correspond exactly to half-complete graphs.

**5.3.2.** *Not too large trees.* The aim of this subsection is to point out that a fair number of trees of different order can be packed into $K_n$ provided the trees are not too large. The following theorem was proved by Bollobás in [7].

THEOREM 5.15. *Suppose that* $3 \le s < (\sqrt{2}/2)n$ *and* $T_2, T_3, \ldots, T_s$ *are trees such that* $T_i$ *has order* $i$ *for each* $i$. *Then for each* $k$, $2 \le k < s$, *every packing of* $T_{k+1}, T_{k+2}, \ldots, T_s$ *into* $K_n$ *can be extended to a packing of* $T_k, T_{k+1}, \ldots, T_s$ *into* $K_n$. *In particular*, $T_2, T_3, \ldots$ $\ldots, T_s$ *can be packed into* $K_n$.

Proof. Consider a packing of $T_{k+1}, T_{k+2}, \ldots, T_s$ into $K_n$ and put $H = G - \bigcup_{j=k+1}^{s} T_j$. The graph $H$ has order $n$ and size

$$e(H) = \binom{n}{2} - \sum_{j=k+1}^{s} (j-1) = \frac{1}{2}(n^2 - n - (s+k-1)(s-k)). \tag{1}$$

The graph $H$ has a subgraph $F$ of minimum degree at least $k-1$ since otherwise

$$e(H) \le \binom{k-1}{2} + (k-2)(n-k+1). \tag{2}$$

Relations (1) and (2) imply

$$2k^2 - 2k(n+2) + n^2 + 3n - s^2 + s \le 0,$$

which is false since $(n+2)^2 < 2(n^2 + 3n - s^2 + s)$.

Finally, by the "folklore lemma" (Theorem 3.37), $\delta(F) \ge k$ implies that $F$ has a subgraph isomorphic to $T_k$. ■

Remark. As was observed in [7], one can improve the above theorem if the Erdős–Sós Conjecture (see page 31) is true since then the bound $(\sqrt{2}/2)n$ in the theorem could be replaced by $(\sqrt{3}/2)n$, which would be essentially better.

Using the same idea as in previous theorem, Y. Caro and Y. Roditty [18] obtained some related results.

**5.4. Bipartite graphs.** The problem of the packing of graphs into the complete bipartite graph $K_{p,q}$ (instead of $K_n$) has appeared in connection with some decomposition problems analogous to the TPC considered in the previous section. We have already seen a theorem on the decomposition of the complete bipartite graph $K_{n,n}$ into a sequence of paths $P_3, \ldots, P_{2n-1}$ of even lengths. Also the packing of sequences of trees into $K_{p,q}$ with $p = n$ or $p = n-1$ and $q = n/2$ or $q = n/2$ has been considered (see for instance [43], [44]).

In this section we shall give without proof some results concerning the packing into bipartite graphs analogous to the "classical" theorems on packing into complete graphs.

To formulate the results we shall need some additional definitions. Let $G = (L, R, E)$ be a bipartite graph with colour classes $L$, $R$ and edge set $E$. This means that $L$ and $R$ are two disjoint sets of independent vertices of the graph $G$ such that $L \cup R = V(G)$. We call $L = L(G)$ and $R = R(G)$ the *left* and the *right set of bipartition* and we denote by $\Delta_L(G)$ and $\Delta_R(G)$ the maximum vertex degree in the set $L$ and $R$, respectively. If $|L| = p$ and $|R| = q$, we say that $G$ is a $(p,q)$-*bipartite* graph. We also say that $G$ admits a $(p,q)$ bipartition. Then $G$ can be considered as a subgraph of the complete bipartite graph $K_{p,q}$.

Let $G = (L, R, E)$ and $H = (L', R', E')$ be two $(p,q)$-bipartite graphs. We say that $G$ and $H$ are *packable into* $K_{p,q}$ (or *with respect to the partition* $(p,q)$) if there is a bijection

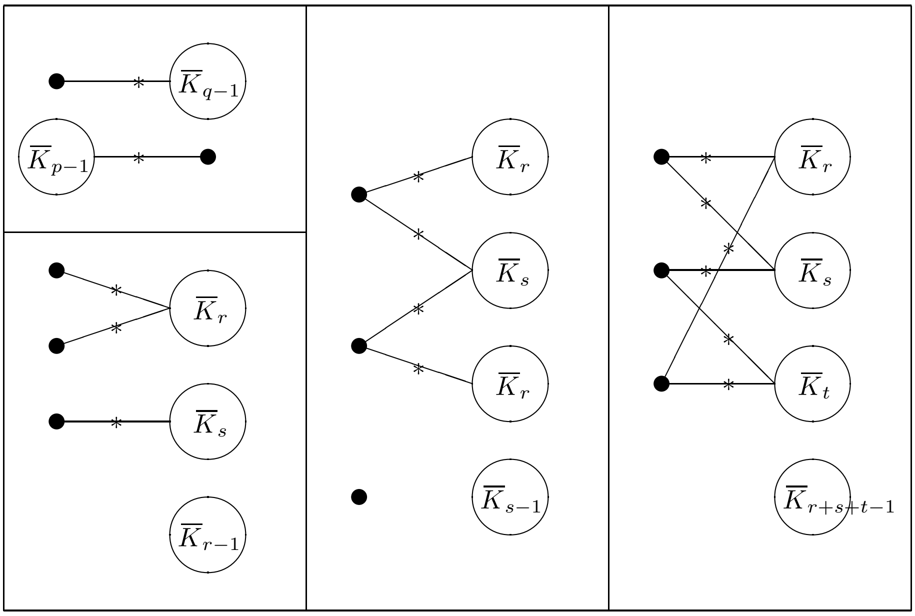



Fig. 5.9. Families of $(p, q)$-bipartite graphs of size $p+q-2$ which have no 2-placement into $K_{p,q}$ (an "edge" marked by $*$ stands for all edges between two sets of vertices).

$\phi : L \cup R \to L' \cup R'$ such that $\phi(L) = L'$ and for every edge $xy \in E$, $\phi(x)\phi(y) \notin E'$.

The function $\phi$ is called the packing of $G$ and $H$ (into $K_{p,q}$). As in the case of packing into the complete graph, the packing of two copies of $G$ into $K_{p,q}$ is called a 2-*placement* (into $K_{p,q}$).

Let us give some examples. We begin with a theorem proved in [34]. This theorem is in a sense analogous to Theorem 2.1.

THEOREM 5.16. *Let $G$ be a $(p, q)$-bipartite graph of size at most $p + q - 2$, $p, q \geq 2$. Then either $G$ is 2-placeable into $K_{p,q}$ or $e(G) = p + q - 2$ and $G$ is one of the graphs of the families given in Fig.* 5.9. ■

Some other related results are given in [34] (see also [74] and [50]).

There is another "natural" possibility of defining a packing for bipartite graphs. Observe that a nonconnected bipartite graph may have several partitions. So, it may happen that for a given bipartite graph, the existence of a 2-placement depends of the partition chosen. For instance, the graph $2S_3$ is 2-placeable in $K_{2,4}$ but not in $K_{3,3}$ (cf. Fig. 5.10). We say that two bipartite graphs $G$ and $H$ are *compatible* if for some integers $p, q$ both $G$ and $H$ admit a $(p, q)$ bipartition. There is a *bipartite packing* of two bipartite graphs $G$ and $H$ if there exists a common bipartition $(p, q)$ such that $G$ and $H$ are packable in $K_{p,q}$.

As an example of a result in this direction, we mention an immediate consequence of Theorem 5.16 which is analogous to the basic theorem for complete graphs.

Fig. 5.10. Two different partitions of $V(2S_3)$

Theorem 5.17. *Let $G$ be a bipartite graph of order $n$ and size at most $n-2$. Then there exists a bipartite packing of two copies of the graph $G$.* ∎

Unfortunately, a similar conclusion does not hold for two nonisomorphic graphs, even if we assume that they are forests. For instance, the forest $F_1 = 2K_{1,2}$ is compatible with the forest $F_2 = K_{1,1} \cup K_{1,3}$ but they are not packable in $K_{2,4}$. This last example is just the smalest one of a large family given by Wang in [69], where he proved that:

Theorem 5.18. *Let $F_1$ and $F_2$ be two compatible forests. Then there exists a bipartite packing of $F_1 \cup K_1$ and $F_2 \cup K_1$.* ∎

We refer the reader to [70] for other results in this direction.

The following theorem was first proved in [20]. We restate it here as a "packing" theorem (cf. [34]).

Theorem 5.19. *Let $G$ and $H$ be two $(p,q)$ bipartite graphs such that*

$$\Delta_L(G)\Delta_R(H) + \Delta_L(H)\Delta_R(G) \leq \max(p,q).$$

*Then $G$ and $H$ are packable into $K_{p,q}$.* ∎

We finally mention that Hajnal and Szegedy obtained in [38] results on packing into bipartite graphs involving average and maximum degree by probabilistic methods.

**5.5. Packing of digraphs.** If we think of the edge between two vertices as an ordered pair rather then a set, a natural direction from the first vertex of the pair to the second can be associated with the edge. Such edges will be called *arcs*, and graphs in which each edge has such a direction will be called *directed graphs* or *digraphs*.

For digraphs, the number of arcs directed away from a vertex $v$ is called the *outdegree* of $v$ and the number of arcs directed into a vertex $v$ is called the *indegree* of $v$.

The vertex set and the arc set of a directed graph $D$ are denoted by $V(D)$ and $A(D)$, respectively. The complete symmetric digraph of order $n$ is denoted by $K_n^*$. Observe that $|A(K_n^*)| = n(n-1)$.

$\overline{D}$ denotes the complement of $D$ with respect to the complete symmetric of order $|V(D)|$. A digraph is said to be *self-complementary* if it is isomorphic to its complement.

In the figures we adopt the convention explained in Fig. 5.11.

Let $D$ and $D'$ be digraphs of order $n$. A *packing* of $D$ and $D'$ (into $K_n^*$) is a bijection $\sigma : V(D) \to V(D')$ such that $(\sigma(x), \sigma(y)) \notin A(D')$ whenever $(x,y) \in A(D)$.

Thus a packing of $D$ and $D'$ exists if and only if $D'$ is contained in $\overline{D}$. If there exists a packing of $D$ and $D'$, then we say that $D$ and $D'$ are *packable*.

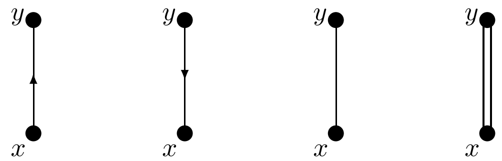



Fig. 5.11. arc $xy$, arc $yx$, arc $xy$ or $yx$, arc $xy$ and $yx$

A connected digraph $D$ with $|V(D)| = |A(D)|$ in which every vertex has outdegree 1 will be called an *insun*. Similarly, an *outsun* is a connected digraph $D$ with $|V(D)| = |A(D)|$ in which every vertex has indegree 1. An example of an outsun is given in Fig. 5.12.

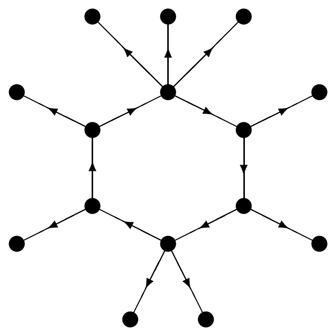


Fig. 5.12. An example of an outsun

The following theorem was proved in [3].

THEOREM 5.20. *If $D$ and $D'$ are digraphs of order $n$ with $|A(D)| \times |A(D')| < n(n-1)$, then there is a packing of $D$ and $D'$.*

Proof. Let $D, D'$ be digraphs satisfying the conditions of the theorem and let $\sigma$ be a random bijection from $V(D)$ to $V(D')$. For $a = (x, y) \in A(D)$ and $a' = (x', y') \in A(D')$ we denote by $\mathcal{A}_{aa'}$ the event that $\sigma(x) = x'$ and $\sigma(y) = y'$. Then

$$\Pr(\mathcal{A}_{aa'}) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

and

$$\Pr(\sigma \text{ is not a packing}) = \Pr\Big(\bigcup_{\substack{a \in A(D) \\ a' \in A(D')}} (\mathcal{A}_{aa'})\Big) \le \sum_{\substack{a \in A(D) \\ a' \in A(D')}} \Pr(\mathcal{A}_{aa'}) = \frac{|A(D)| \cdot |A(D')|}{n(n-1)} < 1.$$

It follows that some bijection from $V(D)$ to $V(D')$ must be a packing of $D$ and $D'$. ■

COROLLARY 5.21. *If $D$ and $D'$ are digraphs of order $n$ with $|A(D)| + |A(D')| \le 2n-2$, then there is a packing of $D$ and $D'$.*

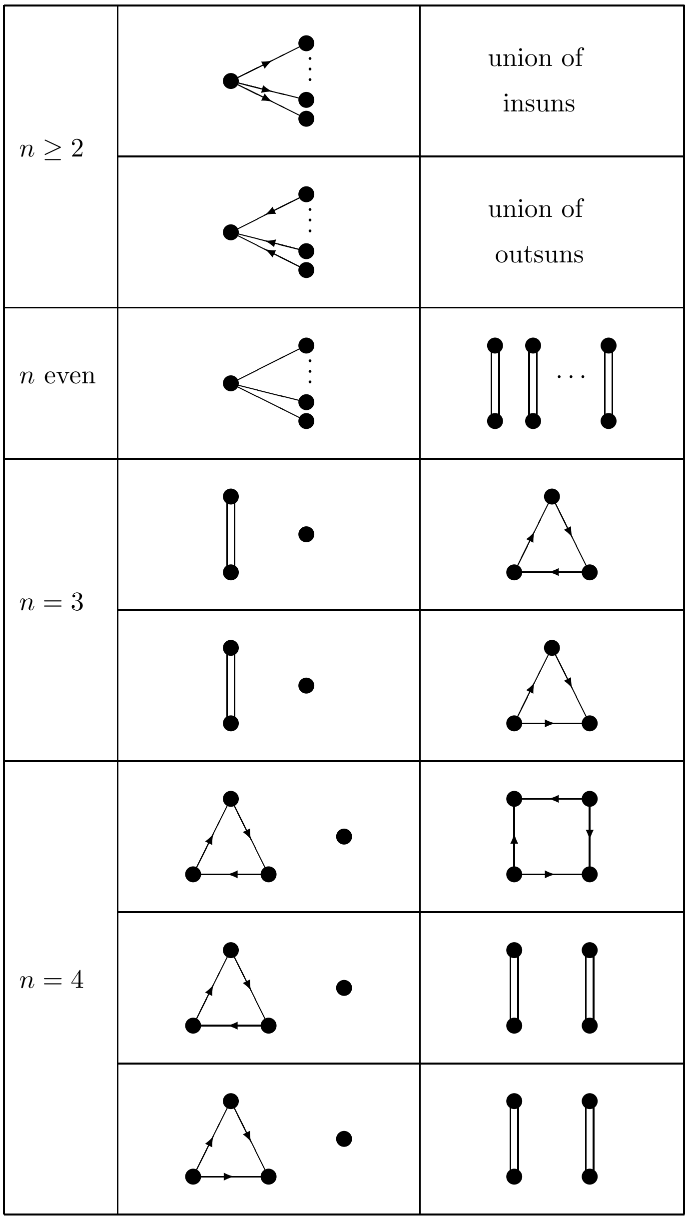



Fig. 5.13. Non-packable pairs of digraphs of order $n$ with the sum of sizes $\leq 2n - 1$

P r o o f. Given two digraphs such that $|A(D)|+|A(D')| = 2n-2$, the number $|A(D)| \cdot |A(D')|$ is maximal if $|A(D)| = |A(D')| = n - 1$. Noting that $(n - 1)^2 < n(n - 1)$ for $n > 1$, we can apply the preceding theorem. ∎

Corollary 5.22. *If $D$ and $D'$ are digraphs of order $n$ such that $|A(D)| \geq (n - 1)^2$ and $|A(D')| \leq n - 1$, then $D'$ is contained in $D$.* ∎

COROLLARY 5.23. *If $D$ and $D'$ are digraphs of order $n$ such that $|A(D)| > (n-1)^2$ and $|A(D')| \le n$, then $D'$ is contained in $D$.* ■

In particular, the last corollary implies Lewin's result that every digraph of order $n$ and size greater than $(n-1)^2$ is hamiltonian.

The authors of [3] also characterized all pairs $D, D'$ of digraphs of order $n$ such that $|A(D)| + |A(D')| = 2n - 1$ and there is no packing of $D$ and $D'$. We state here this theorem without proof.

THEOREM 5.24. *Let $D$ and $D'$ be digraphs of order $n$ such that $|A(D)| \le |A(D')|$ and $|A(D)| + |A(D')| = 2n - 1$. Then there is no packing of $D$ and $D'$ if and only if $(D, D')$ is one of the pairs listed in the table in Fig. 5.13.* ■

We finish this section by mentioning two open problems. The first one concerns the packing of two copies of a digraph and was first formulated in [4]. We state it here in a "without exceptions" version.

CONJECTURE 5.25. *Every digraph of order $n$ and size at most $2n - 4$ is contained in a self-complementary digraph.*

Clearly, if a digraph $D$ is contained in a self-complementary digraph, then $D$ is embeddable in its complement, i.e. two copies of $D$ are packable. Recently, the following partial result was announced in [76]:

THEOREM 5.26. *Let $D$ be a digraph of order $n$. If $|A(D)| \le 3(n-2)/2$, then $D$ is contained in a self-complementary digraph.* ■

The second problem was posed by Wojda in [73]. For every $n$ and $k$, $1 \le k \le n(n-1)$, denote by $f(n, k)$ the smallest number such that there exist digraphs $D_1$ and $D_2$ with $|A(D_1)| = k$ and $|A(D_2)| = f(n, k)$ for which there is no packing of $D_1$ and $D_2$.

CONJECTURE 5.27. *For every $m$ satisfying $2 \le m \le n/2$,*

$$f(n, n-m) = 2n - \left\lfloor \frac{n}{m} \right\rfloor.$$

Two particular cases of this conjecture, $m = 2$ and $m = n/2$, were proved by the authors of [75], but the proofs are very long and have not been published yet.

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