On directed versions of 1-2-3 Conjecture

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(Kyoto, 2016)

Joint work with

Mirko Horňák (UPJS, Košice, Slovakia) and

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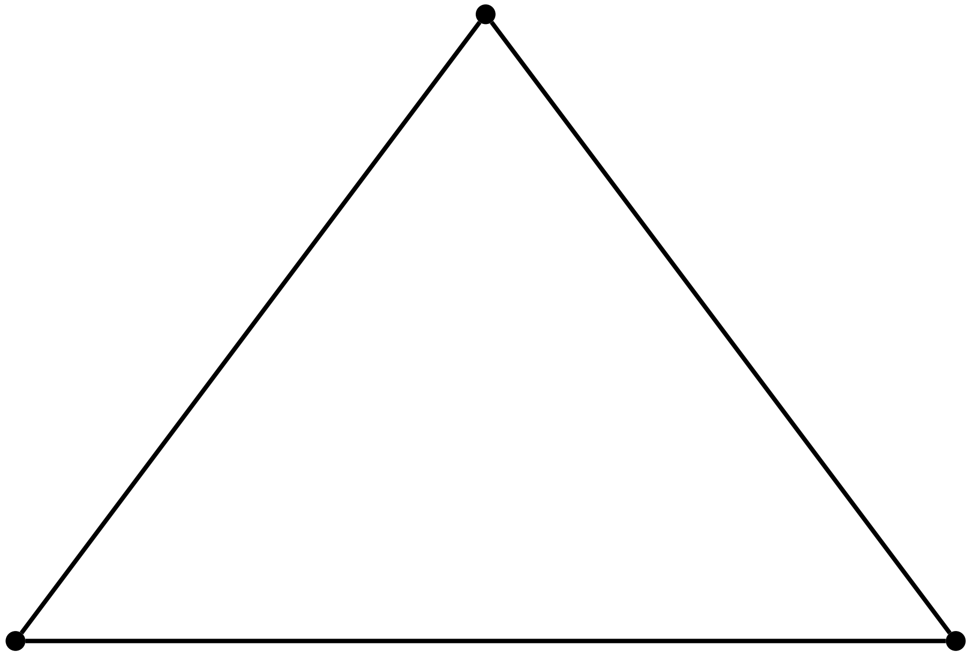
Mirko Horňák (UPJS, Košice, Slovakia) and

Jakub Przybyło (AGH University, Cracow, Poland)

Motivation



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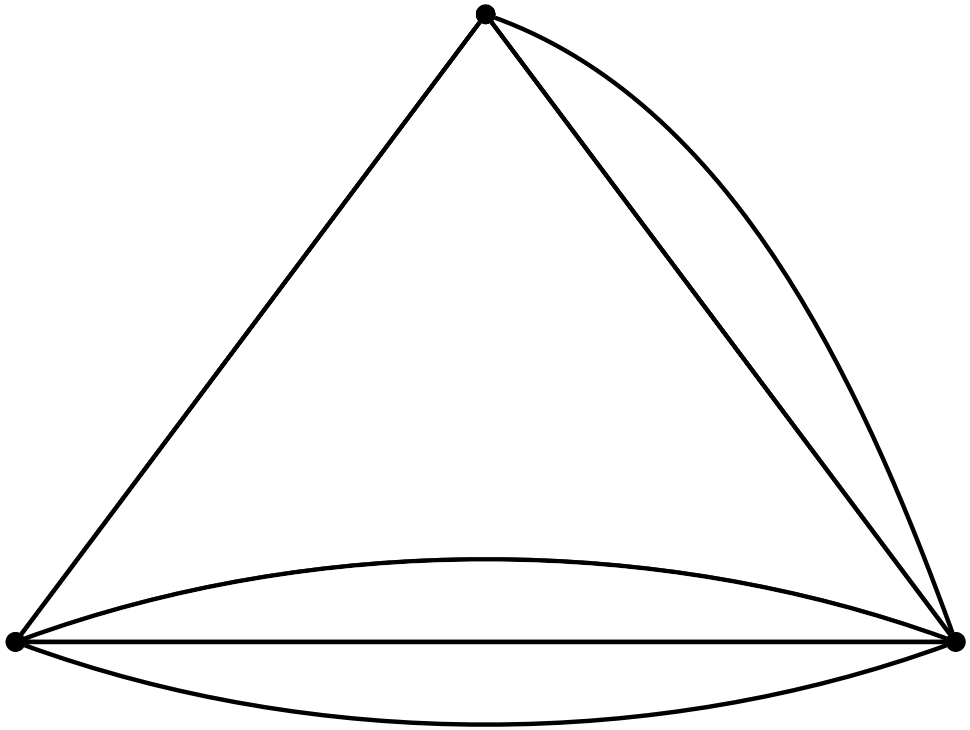

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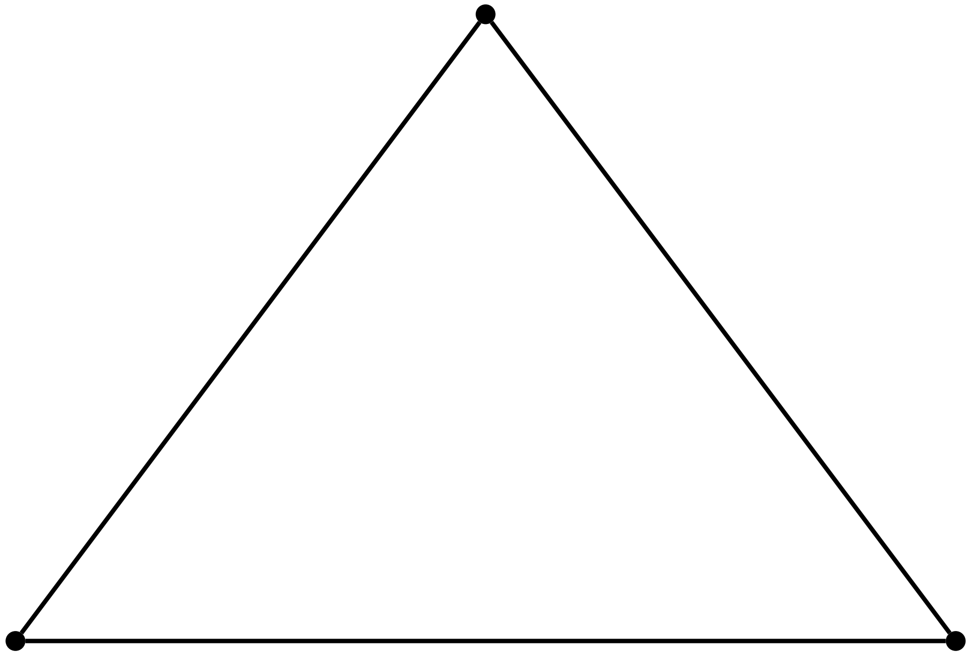
- and is still intensely studied.

Irregularity strength and coloring



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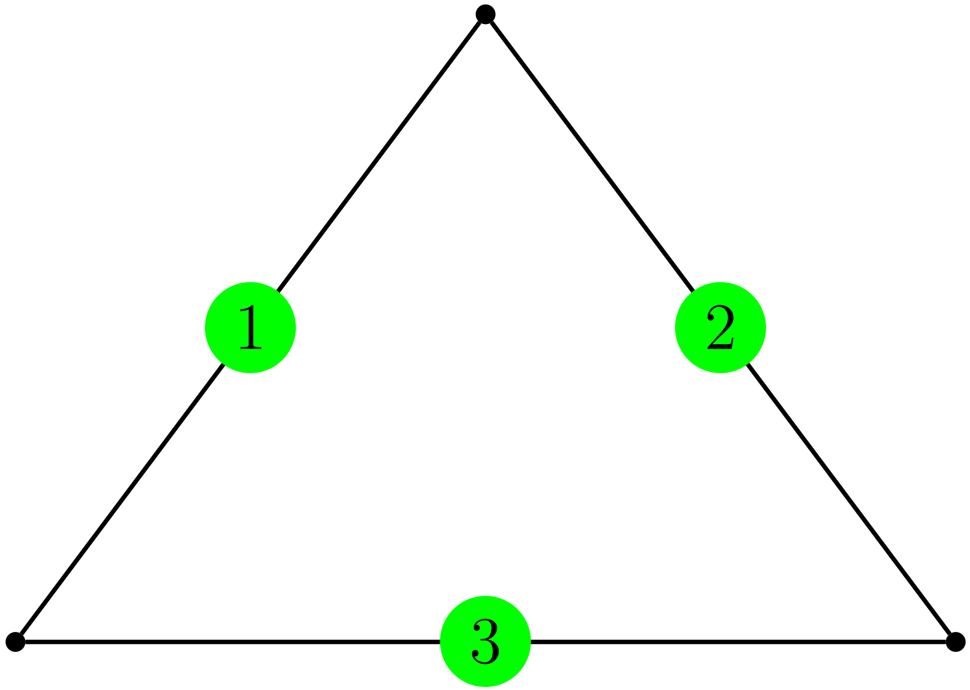

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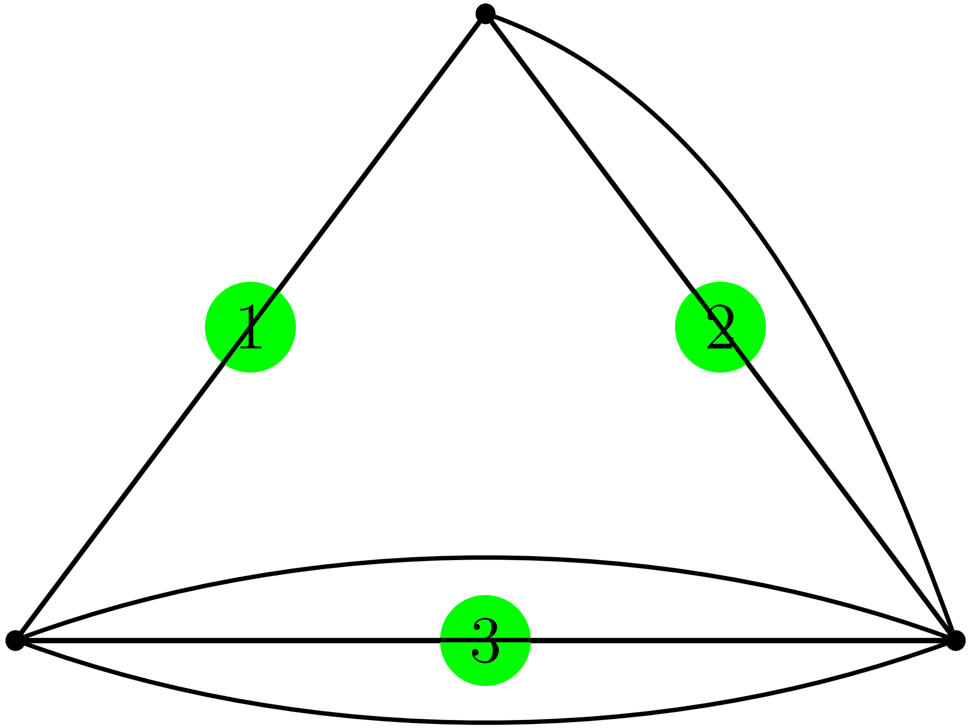

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- irregularity strength is minimum k such that there exists an f distinguishing all vertices.

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- $\chi_\sigma \leq 3$,

Local version. What is known?

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- $\chi_\sigma \leq 5$
  (M. Kalkowski, M. Karoński, F. Pfender; 2011)

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- In order to distinguish two vertices x, y we can use σ^+ and σ^-.

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- M. Borowiecki, J. Grytczuk, M. Pilśniak. Coloring chip configurations on graphs and digraphs. *Information Processing Letters*, 112:1-4, 2012.

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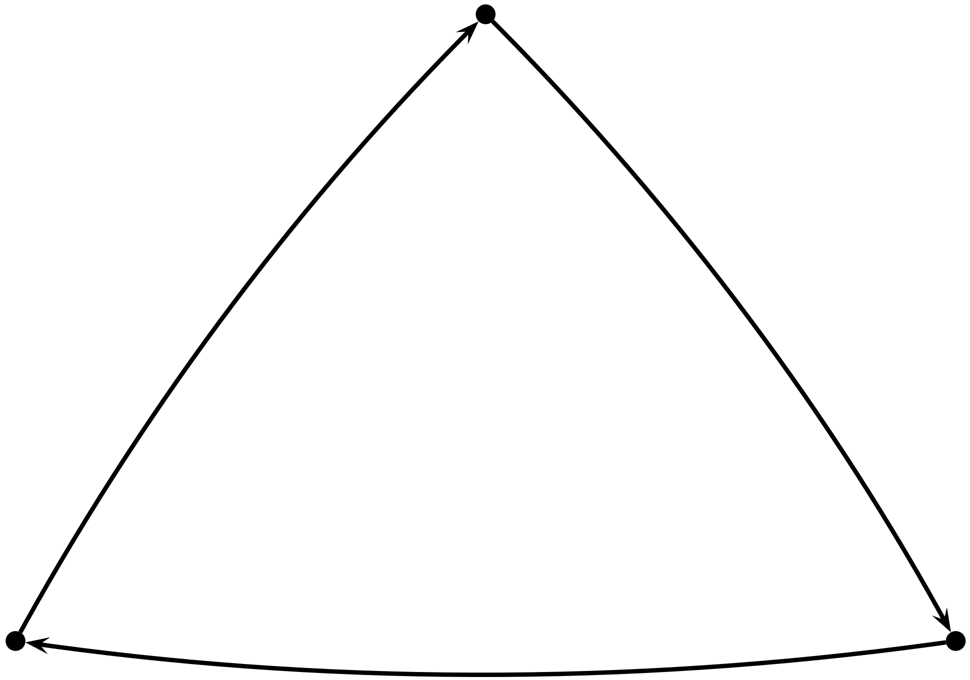
- O. Baudon, J. Bensmail, É. Sopena. An oriented version of the 1-2-3 Conjecture. *Discussiones Mathematicae Graph Theory*, 35(1):141-156, 2015

We need 3 colors



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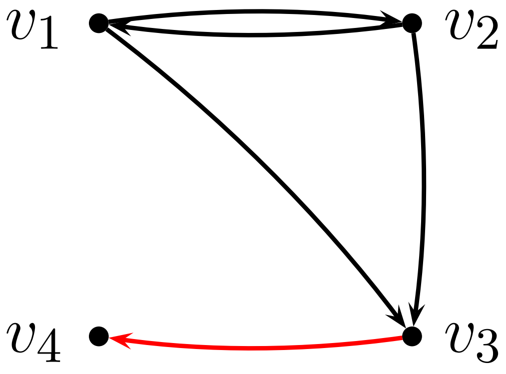
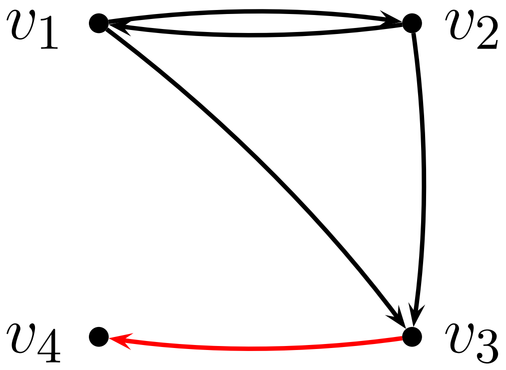
- Unfortunately, such coloring is not always possible

Lonely arcs



$v_3 v_4 \in A$;

Lonely arcs



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$d^+(v_3) = 1 \Rightarrow \sigma^+(v_3) = f(v_3v_4)$

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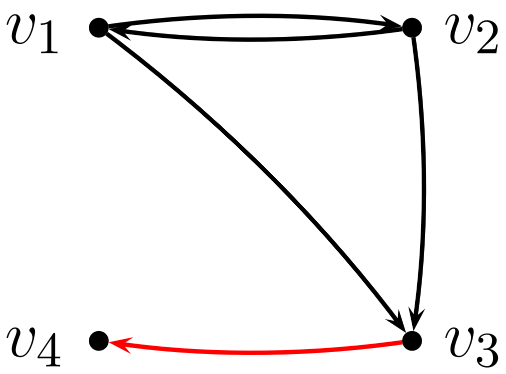

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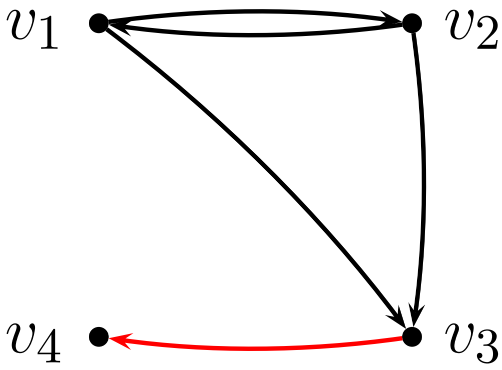

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So, it is impossible to distinguish v_3 from v_4.

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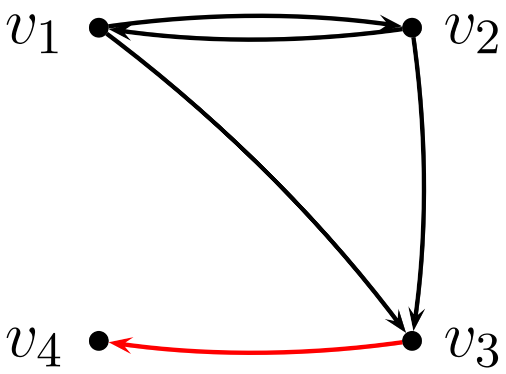

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- such an arc is called **lonely**.

Third possibility. The main theorem

- Theorem. Let $D = (V, A)$ be a digraph without lonely arcs.

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- Emma Barme, Julien Bensmail, Jakub Przybyło, Mariusz Woźniak, On a directed variation of the 1-2-3 and 1-2 Conjectures, submitted.

Fourth possibility

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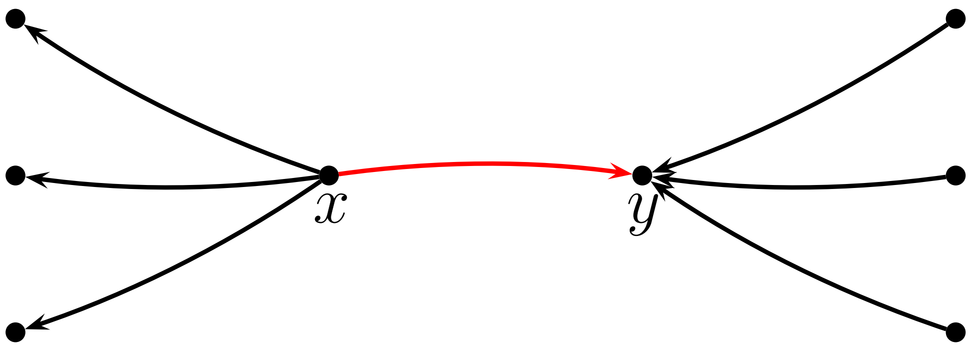
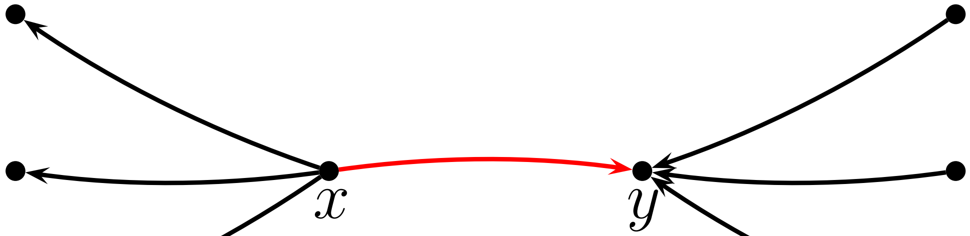
- As in Łuczak’s problem, such coloring is not always possible

Source-sink arcs



x is a source, y is a sink,

Source-sink arcs



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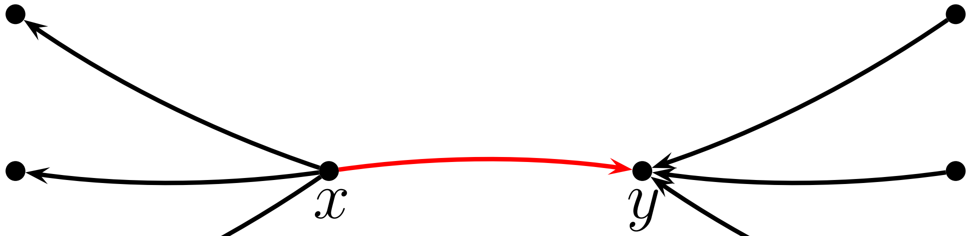

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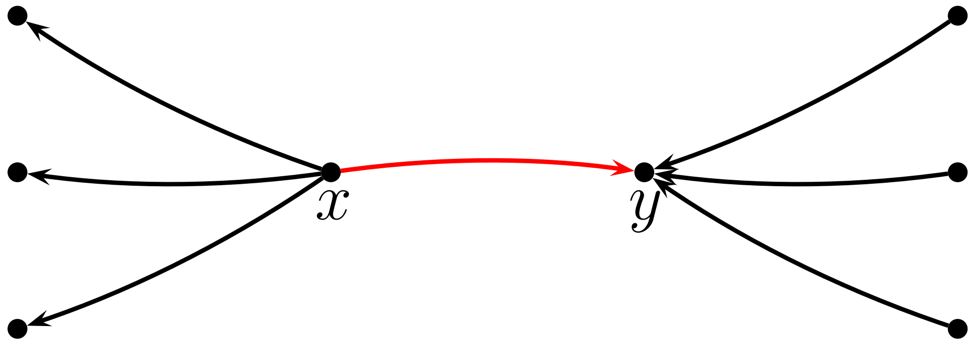

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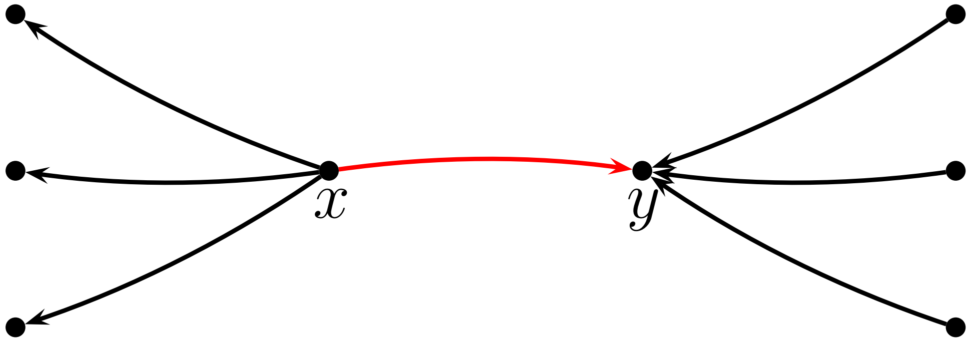

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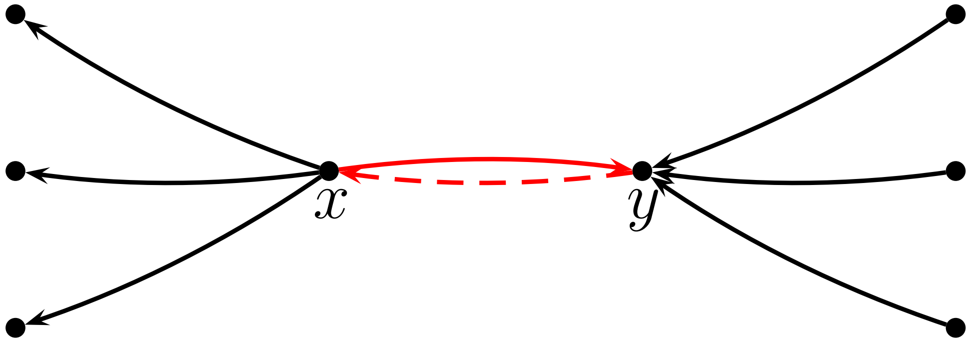
- then, the arc xy is called **source-sink arc**.

Source-sink edge



If x is a source, y is a sink in the graph without yx arc,

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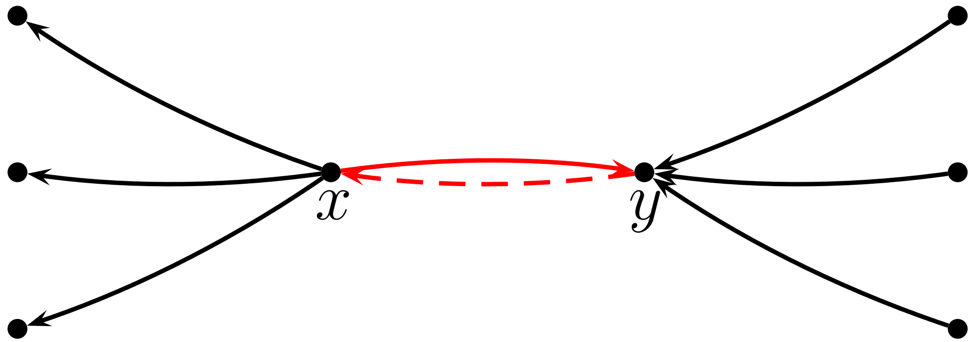

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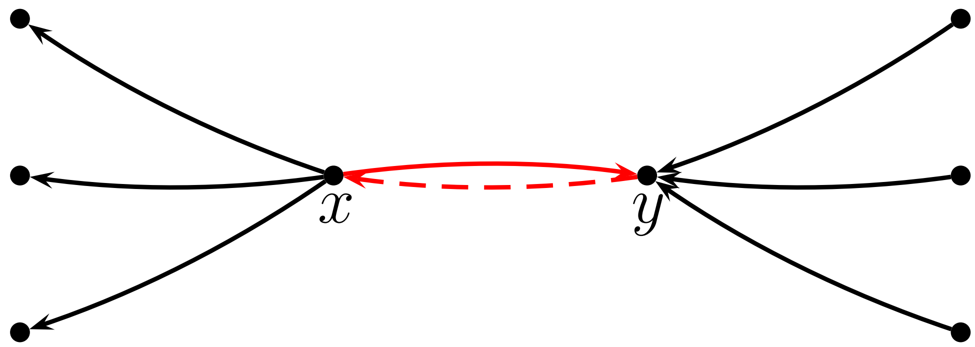

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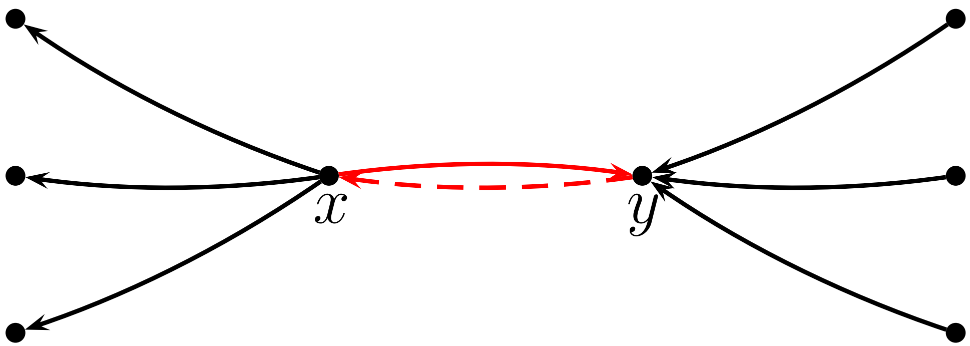

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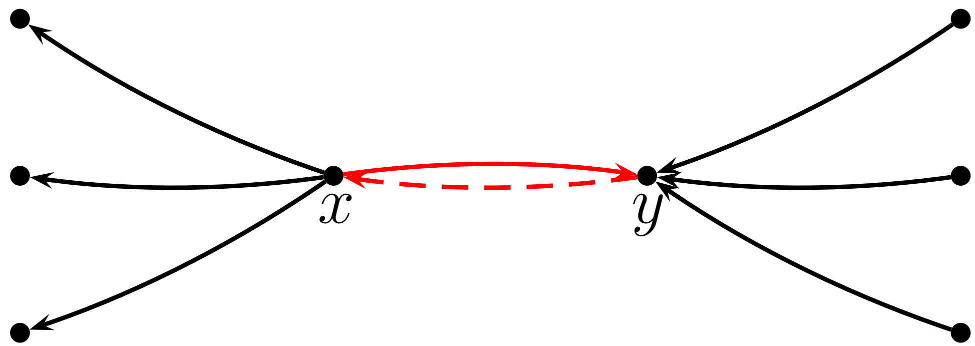

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- in this case, the arc xy is called **source-sink edge**.

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- Theorem. Let $D = (V, A)$ be a digraph without source-sink configurations (arcs or edges).

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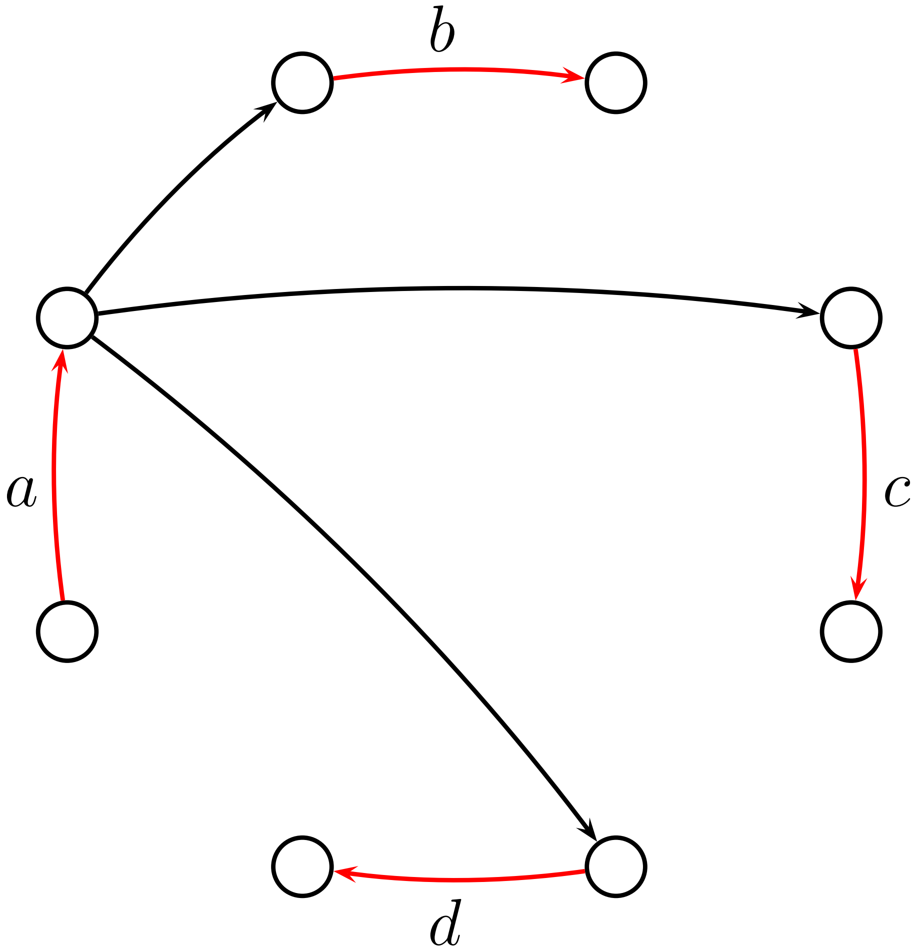
- Answer: no!

An example



- $a \neq b$;

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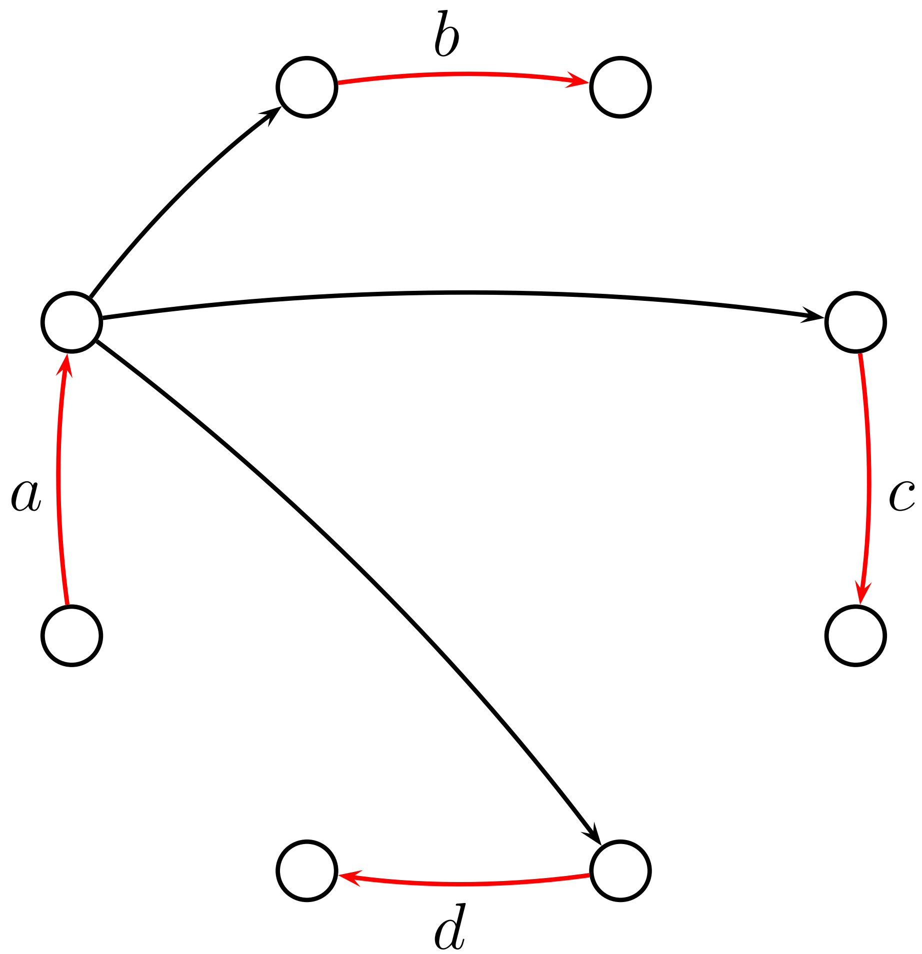

- $a \neq b$;
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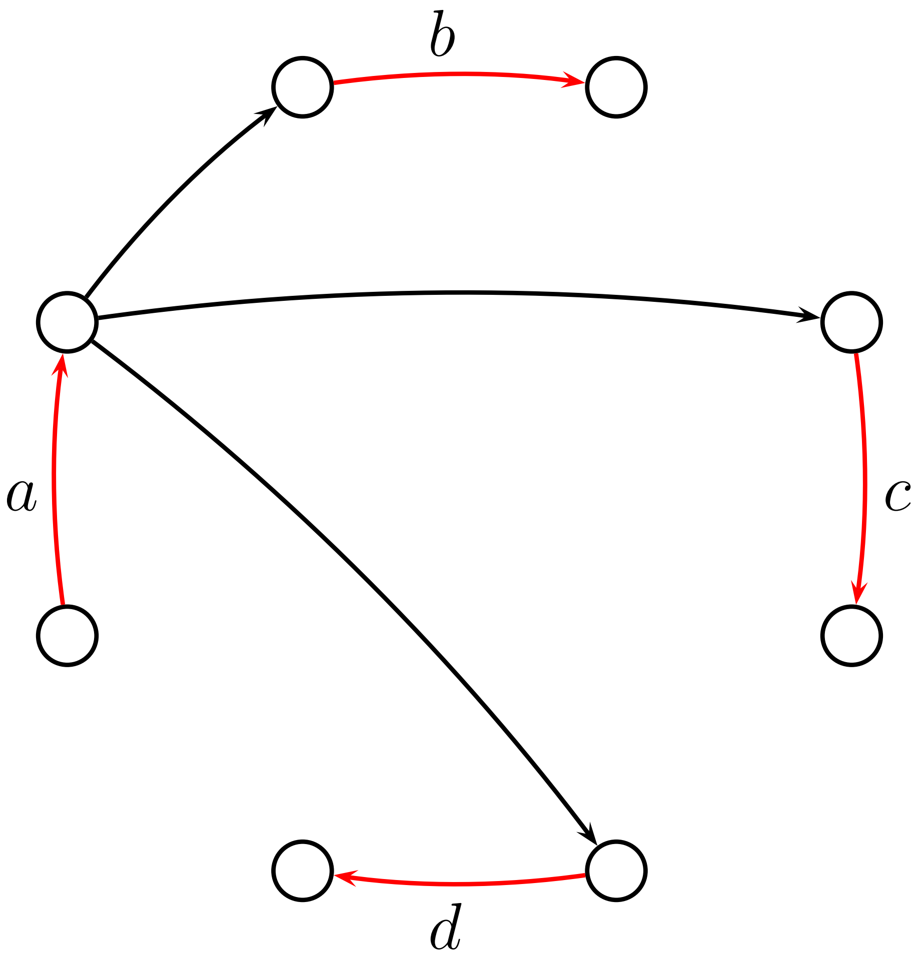

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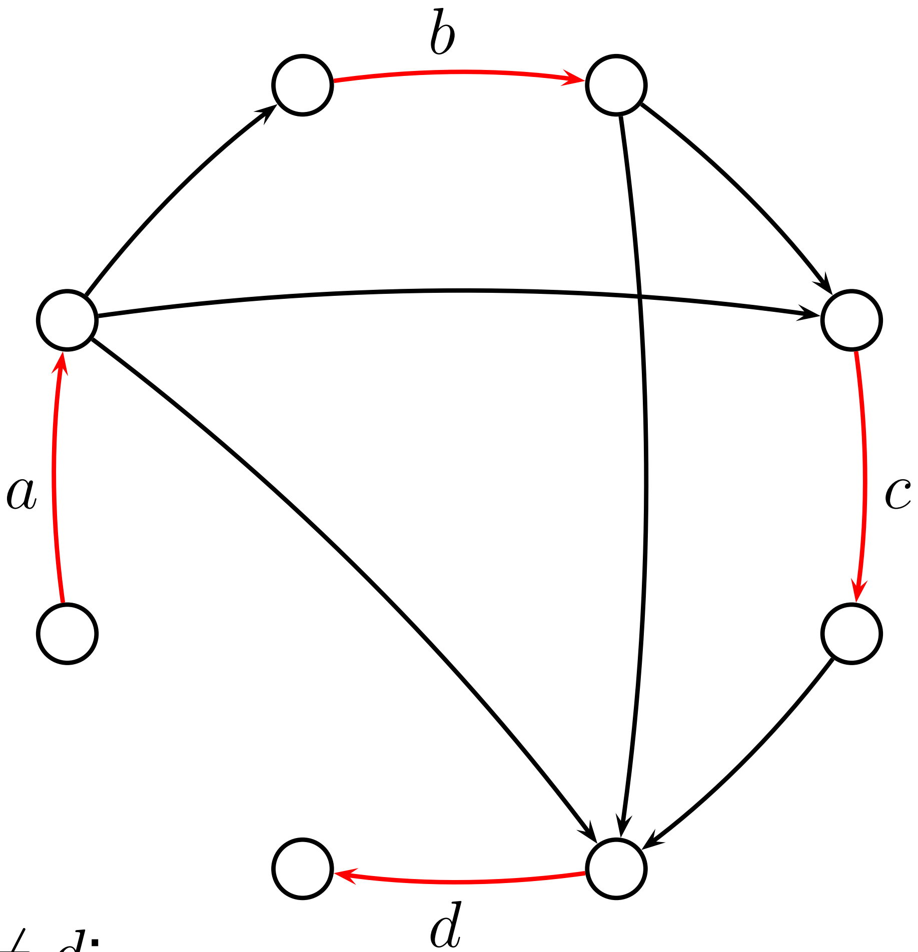

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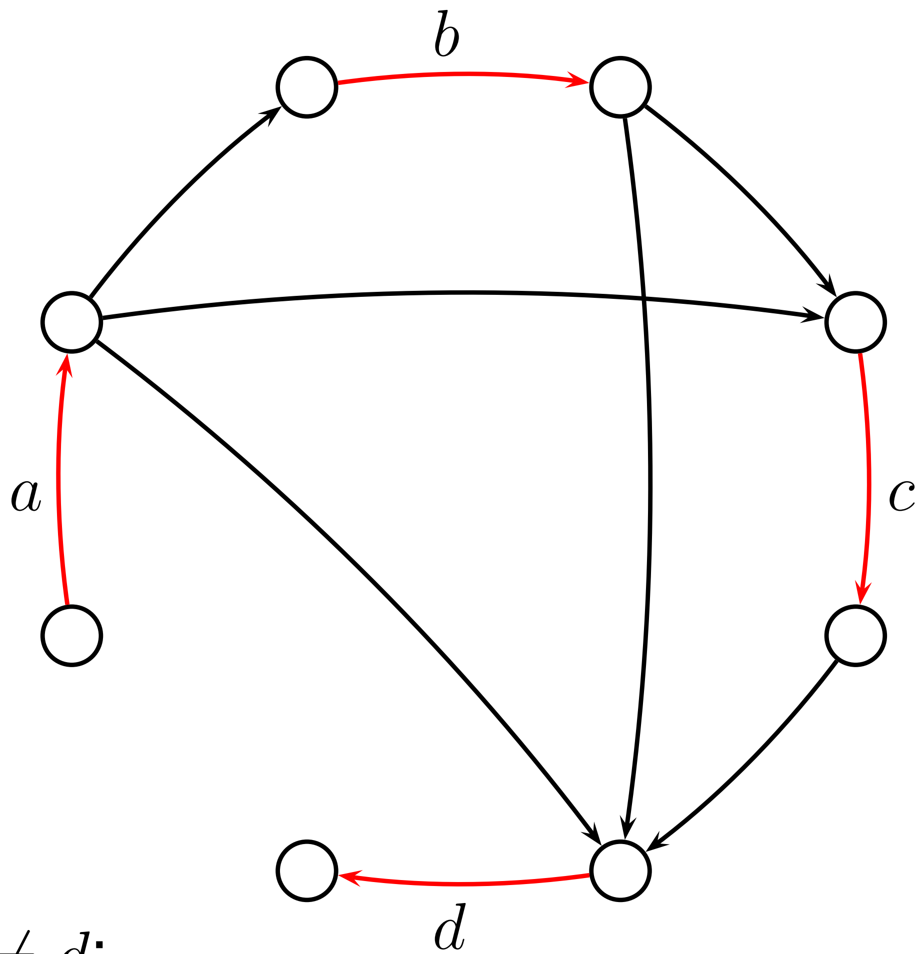

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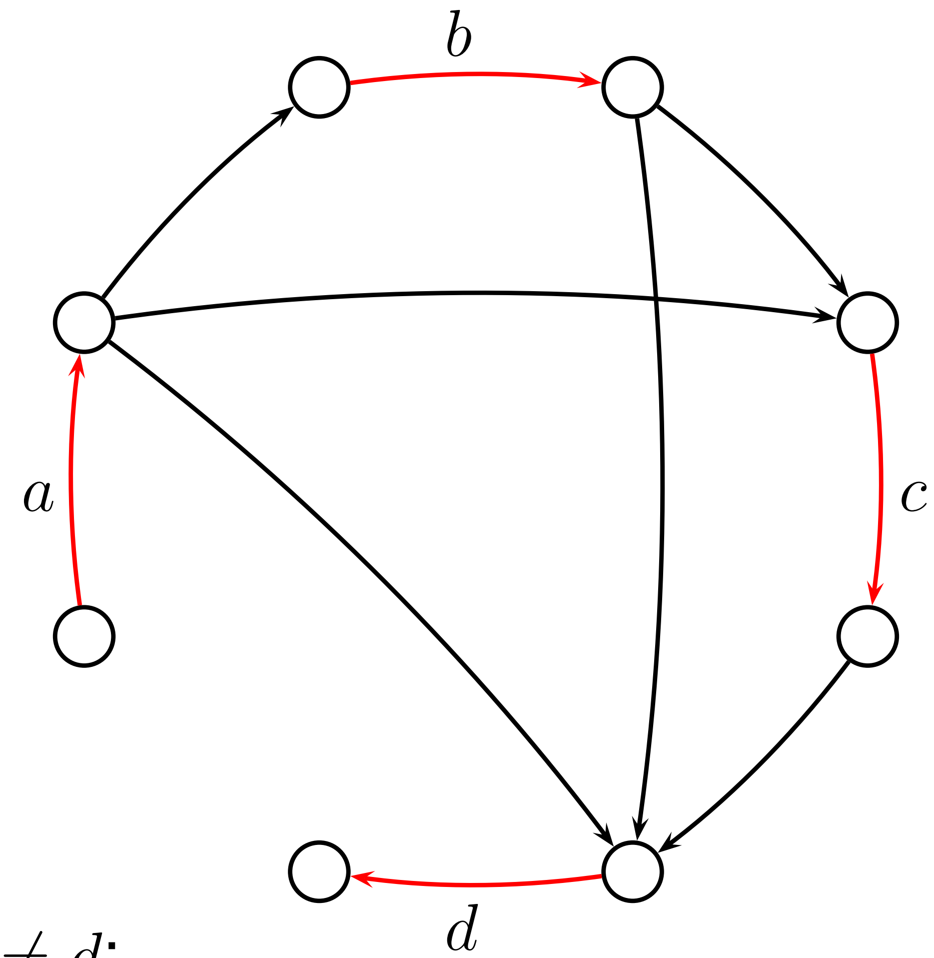

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- In general, for digraphs without source-sink configurations, we have showed that $\overleftarrow{\chi}_L$ is not bounded.
- ... because of lonely edges ....
- So, maybe without such edges ... ?

A conjecture

Conjecture. Let $D = (V, A)$ be a digraph without source-sink configurations

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- For the moment, we are able to prove 4.

The end

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