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Packing of graphs and permutations—a survey

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Abstract

An embedding of a graph G (into its complement $\bar{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. If there exists an embedding of G we say that G is *embeddable* or that there is *a packing* of two copies of the graph G into complete graph K_n. In this paper we discuss a variety of results, some quite recent, concerning the relationships between the embeddings of graphs in their complements and the structure of the embedding permutations.

Keywords: Packing of graphs; Self-complementary graphs; Permutation (structure)

1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs $G=(V(G),E(G))$ of order $n=|V(G)|$ and size $|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges. If a graph G has order n and size m, we say that G is an (n,m)-graph. An *embedding* of G (in its complement $\bar{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In others words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G (of order n) into the complete graph K_n.

The following theorem was proved, independently, in [2,5,23].

Theorem 1. *Let $G=(V,E)$ be a graph of order n. If $|E(G)| \leqslant n-2$ then G can be embedded in its complement.*

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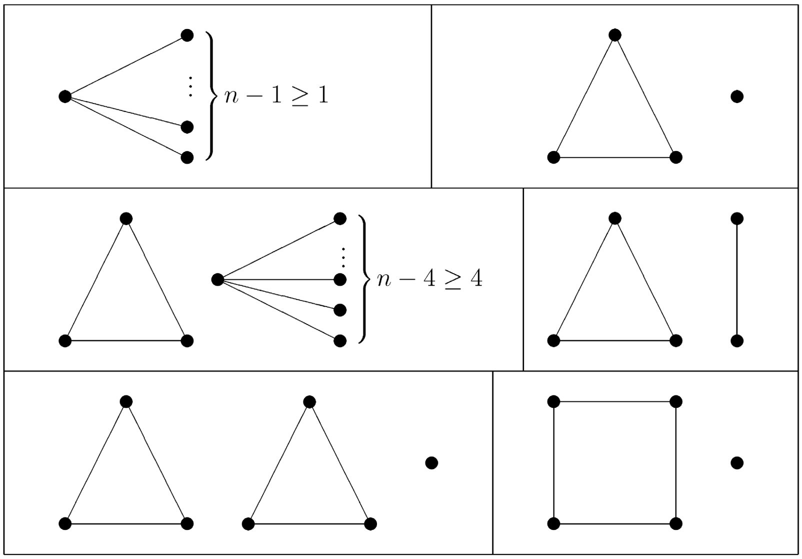


Fig. 1. Non-embeddable $(n, n-1)$ graphs.

The example of the star S_n shows that Theorem 1 cannot be improved by raising the size of G even in the case when G is a tree. However, in this case we have the following theorem, proved in [6] and illustrated by Fig. 1, which completely characterizes those graphs with n vertices and $n-1$ edges that are embeddable.

Theorem 2. *Let $G=(V,E)$ be a graph of order n. If $|E(G)| \leqslant n-1$ then either G is embeddable or G is isomorphic to one of the following graphs*: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ *with* $n \geqslant 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.

The general result known on embeddings of (n,n)-graphs is the following theorem proved in [9].

Theorem 3. *Let $G=(V,E)$ be a graph of order n If $|E(G)|=n$ then either G is embeddable or G is isomorphic to one of the graphs of* Fig. 2.

These results has been improved in many ways. In this survey, we are interested in these improvements of Theorem 1 that deal with some additional properties of packing permutations.

The main references of the paper and of other packing problems are the last chapter of Bollobás's book [2], the fourth Chapter of Yap's book [35] and the survey papers [31,36].

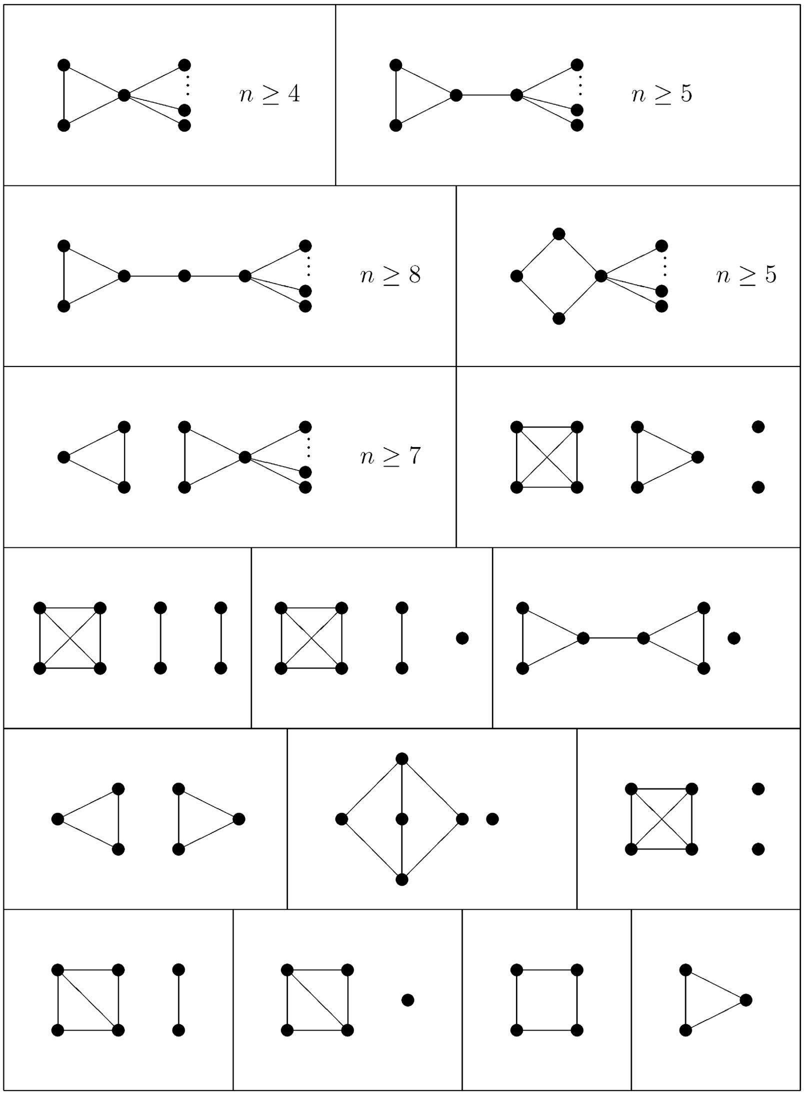


Fig. 2. Non-embeddable (n,n) graphs.

2. Self-complementary graphs

A graph G is *self-complementary* (briefly, s-c) if it is isomorphic to its complement (cf. [20,22], or [12]). It is clear that an s-c graph has $n \equiv 0,1 \pmod 4$ vertices. We extend the above definition to the case where $n \equiv 2,3 \pmod 4$ as follows. A graph G of order $n \equiv 2,3 \pmod 4$ is *almost self-complementary* (or briefly, a-s-c) if G is of size $\frac{1}{2}(\binom{n}{2} - 1)$ and G is a subgraph of its complement (see also [8]).

We also say that there exists a packing of G, G, K_2 (into K_n) or that G is self-complementary in $K_n - e$ (cf. [7])

In this paper, we shall use the term 'self-complementary' also in the case $n \equiv 2, 3 \pmod 4$.

We are interested in s-c graphs because it is evident that subgraphs of s-c graphs are embeddable. It is known that for (n,k)-graphs with $k \leqslant n-1$ the property 'to be embeddable' and the property 'to be a subgraph of a s-c graph' are in fact equivalent (see [1,29]). In all cases, the proofs use the cyclic structure of the packing permutation. The case of (n,n)-graphs is considered in [27]. Also in this case, the proof uses the cyclic structure of the packing permutation.

The following theorem characterizes the structure of m s-c permutation that is a permutation which transforms one copy of an s-c graph into another. The part concerning the cases $n \equiv 0, 1 \pmod 4$ was proved in [20,22]. The part concerning the cases $n \equiv 2, 3 \pmod 4$ was proved in [8].

Theorem 4. *Let $G=(V,E)$ be an s-c graph of order n, and let σ be an s-c permutation of G. Then*

when $n \equiv 0 \pmod 4$, *σ consists of cycles of lengths that are multiples of* 4,

when $n \equiv 1 \pmod 4$, *σ consists of cycles of lengths that are multiples of* 4, *except for one cycle of length one*.

when $n \equiv 2 \pmod 4$, *then either*

- *σ has two fixed points and the other cycles have lengths that are multiples of* 4, *or*
- *σ consists of a cycle of length* $4h+2$, $h \geqslant 1$, *and the other cycles have lengths that are multiples of* 4.

when $n \equiv 3 \pmod 4$, *then either*

- *σ consists of a cycle of length* 3 *and the other cycles have lengths that are multiples of* 4, *or*
- *σ consists of a cycle of length* $4h+2$, $h \geqslant 1$, *and the other cycles have lengths that are multiples of* 4, *and one fixed point.*

As observed above, it is evident that subgraphs of s-c graphs are embeddable. In general the converse is not true. For instance, consider the graph on eight vertices (see Fig. 3) with vertex set $V = \{u, v, x_1, x_2, x_3, y_1, y_2, y_3\}$ and edge set $E = \{vx_i, vy_i, x_iy_i\}$, $i = 1, 2, 3$ (see Fig. 3). Thus, the vertex u is an isolated vertex, the vertex v is of degree 6 while the remaining vertices are of degree two. The permutation $(uv)(x_1x_2x_3)$ provides an embedding permutation which is not however an s-c permutation, by the theorem above, because of the transposition (uv). On the other hand, in each embedding permutation the vertex v has to be mapped on u and none of vertices of degree two can be put onto v. Therefore, each embedding permutation contains a transposition in its cycle structure.

It is interesting to note that within (n,n)-graphs which are embeddable there is only one graph that is not a subgraph of an s-c graph: the graph F_0 drawn in Fig. 4. Let

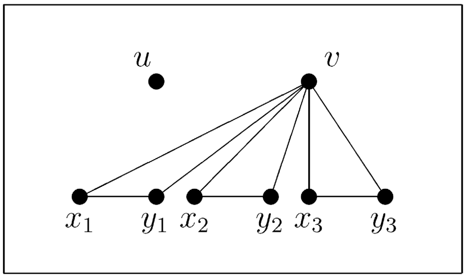


Fig. 3. An embeddable graph without s-c permutation.

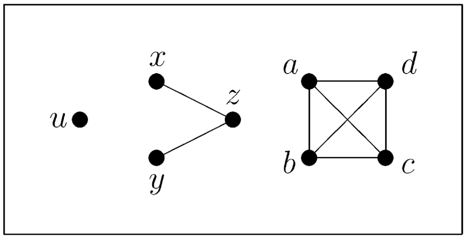


Fig. 4. The exceptional graph F_0.

σ be an embedding of F_0. It is easy to see that the set of the images of the vertices $\{a,b,c,d\}$ of K_4 must contain the vertices u, x, y and one of the vertices of K_4. In this situation, the vertex z has to be mapped on itself, *i.e.* σ must have a fixed point, and therefore cannot be an s-c permutation.

So, we have the following theorem proved in [27].

Theorem 5. *Every embeddable graph of order n and size at most n is a subgraph of an s-c graph of order n except for the graph F_0 defined in Fig.* 4.

3. Three copies of a graph

By analogy with the definitions of an s-c (a-s-c) graph, a graph G of order $n \equiv 0,1 \pmod 3$ is a 3-*s-c graph* if G is of size $\frac{1}{3}\binom{n}{2}$ and the complete graph K_n can be decomposed into three edge-disjoint graphs each of them isomorphic to G.

A graph G of order $n \equiv 2 \pmod 3$ is a *3-a-s-c graph* if G is of size $\frac{1}{3}(\binom{n}{2}-1)$ and the graph $K_n - e$ can be decomposed into *three edge-disjoint* graphs, each of them isomorphic to G (see also [14,25] for some related problems concerning the divisibility of graphs into isomorphic parts).

A packing of three copies of a graph G will be called a 3-*placement* of G. A packing of two copies of G, i.e. a 2-placement, is an *embedding* of G (in its complement $\bar{G}$).

As in the previous section, it is evident that, if G is a subgraph of an 3-self-complementary graph, then there is a 3-placement of G.

We begin with a theorem about 3-placement of a graph, proved in [34], which can be considered as an improvement of Theorem 1.

Theorem 6. *Let $G=(V,E)$ be a graph of order n. If $|E(G)| \leqslant n-2$, then either there exists a 3-placement of G or G is isomorphic to $K_3 \cup 2K_1$ or to $K_4 \cup 4K_1$.*

The following theorem improves Theorem 6 by specifying the permutation structure of the 3-placement of G (see [29]).

Theorem 7. *Let $G=(V,E)$ be a graph of order n, $G \neq K_3 \cup 2K_1$, $G \neq K_4 \cup 4K_1$. If $|E(G)| \leqslant n-2$, then there exists a permutation σ on $V(G)$ such that σ^0, σ^1, σ^2 define a 3-placement of G. Moreover, all cycles of σ have length 3, except for one of length one if $n \equiv 1 \pmod 3$ or two of length one if $n \equiv 2 \pmod 3$.*

We are now going to consider the problem of the 3-placement of a tree T. Observe first that if there is a 3-placement of T in K_n, then we obviously have

$$3(n-1) \leqslant \binom{n}{2},$$

which implies $n \geqslant 6$.

Moreover, since the vertex $v \in V(T)$ such that $d(v)=\Delta(T)$ must be placed with two others vertices with degrees at least one, we must assume that $\Delta(T) \leqslant n-3$.

However, these obvious necessary conditions are not sufficient as shown by the example of S''_6 where, in general, S''_n is a tree obtained from S_{n-2} by inserting two new vertices on one edge (S'_n is a tree obtained from S_{n-1} by inserting one new vertices on an edge). This fact was first observed by Huang and Rosa [16].

Wang and Sauer [26] proved the following theorem.

Theorem 8. *Let T be a tree of order n, $n \geqslant 6$, $T \neq S_n$, $T \neq S'_n$ and $T \neq S''_6$. Then there exists a 3-placement of T.*

An independent proof of Theorem 8 which gives some information about the permutations defining the packing can be found in [31] (in [15] the authors mention another independent proof of this theorem). More precisely we have

Theorem 9. *If $T \neq S''_{3k}$, $k \geqslant 3$, then under the hypotheses of Theorem 8 there exists a permutation σ on $V(T)$ such that $\sigma^0, \sigma^1, \sigma^2$ define a 3-placement of T and σ has all its cycles of length 3, except for one of length 1 if $n \equiv 1 \pmod 3$ or two of length 1 if $n \equiv 2 \pmod 3$,*

If $T = S''_{3k}$, $k \geqslant 3$, then there exists a permutation σ of $V(T)$ such that $\sigma^0, \sigma^1, \sigma^2$ define a 3-placement of T and σ has all its cycles of length 3, except for three cycles of length 1.

4. Cyclically embeddable graphs

If an embedding of G is a cyclic permutation, we say that G is *cyclically embeddable* (CE for short).

Let us start by the following theorem, proved in [24] which has been used in the study of embeddings of $(n, n-1)$ graphs.

Theorem 10. *Let G be a graph of order n. If $|E(G)| \leqslant n-2$ then there exists an embedding σ of G in its complement such that σ has no fixed points, i.e. $\sigma(x) \neq x$ for $x \in V(G)$.*

The above theorem cannot be improved by increasing the number of edges as shown by the graph $K_{1,2} \cup K_3$.

However, Theorem 10 can be improved in an other direction by specifying the structure of the packing permutation. In particular we have the following result proved first in [29].

Theorem 11. *Let G be a graph of order n. If $|E(G)| \leqslant n-2$, then there exists a cyclic embedding of G.*

As we have seen, if $|E(G)| = n-1$, then there are graphs that are not embeddable and even in the case where a graph is embeddable, a fixed-point-embedding does not necessarily exist. However, if we assume in addition that G is a tree, we have the following result (cf. [30]).

Theorem 12. *Let T be a tree of order n. If $T \neq S_n$ then there exists a cyclic embedding of G.*

The main tool in the study of cyclically embeddable graphs is the construction given below. Let G and H be two rooted graphs at u and x, respectively. (By a rooted graph we mean a graph with a specified vertex.) The graph of order $|V(G)| + |V(H)| - 1$ obtained from G and H by identifying u with x will be called the *touch* of G and H and will be denoted by $G \cdot H$. A similar operation consisting in the identification of a couple of vertices of G, say (u_1, u_2), with a couple of vertices of H, say (x_1, x_2) will be called the 2-*touch* of G and H and will be denoted by $G : H$. The graph $G : H$ is of order $|V(G)| + |V(H)| - 2$. By definition, the edge (u_1, u_2) belongs to $E(G : H)$ if $u_1u_2 \in E(G)$ or $x_1x_2 \in E(H)$. (Of course, the definition of the graph $G : H$ depends on (u_1, u_2) and (x_1, x_2).)

Let σ be a cyclic permutation defined on $V(G)$. For $u \in V(G)$, we denote often the vertex $\sigma(u)$ by u^+ and $\sigma^{-1}(u)$ by u^-. If the edge uu^+ belongs to $E(G)$ then it is said to be *of length one* (with respect to σ).

Two lemmas below (see [32]) show how to get a cyclic embedding of a 'bigger' graph having cyclic embeddings of two 'smaller' graphs.

Lemma 13. *Let G and H be two CE graphs rooted at u and x, respectively. Then the graph $G \cdot H$ is CE.*

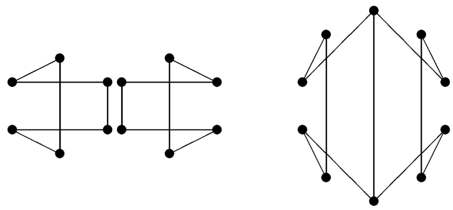


Fig. 5. 2-Touch of two cycles C_6 and the resulting cyclic embedding of C_{10} (with a chord).

Remark. A result holds also if "CE" is replaced by "embeddable" (see [11]).

The proof of the following lemma can be found in [32].

Lemma 14. *Let G and H be two CE graphs such that the vertices v, u of G and x, y of H are consecutive with respect to the cyclic embeddings of G and H, respectively. Suppose that the edges uu^+* and *xx^- as well as the edges yy^+ and vv^- are not simultaneously present. Then the graph $G:H$ obtained by identifying u with x and v with y is CE.*

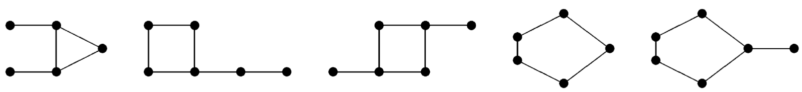
Remark. Observe that the condition at the edges uu^+ and xx^- as well as the edges yy^+ and vv^- are not simultaneously present is in particular fulfilled if uv is an edge of G or xy is an edge of H. Fig. 5 gives an example of an application of Lemma 14.

It is easy to see that neither C_3 nor C_4 are embeddable and that the cycle C_5 is embeddable but not cyclically. However (see [32]):

Theorem 15. *Let C_n be the cycle of order n. If $n \geqslant 6$, then there exists a cyclic embedding of C_n.*

The previous result can be somewhat generalized (see [32]).

Theorem 16. *The unicyclic graphs that are embeddable are also cyclically embeddable except for the five graphs given below.*



Consider now the case of the family of $(n, n-1)$-graphs. As remarked above, the graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$ are embeddable but cannot be embedded without fixed vertices. It is interesting to note that all other $(n, n-1)$-graphs that are contained in their complements can be embedded without fixed vertices. More precisely, we have the following theorem mentioned first in [24].

Theorem 17. *Let G be a graph of order n with $|E(G)| \leqslant n-1$ and such that*

(a) *G is not an exceptional graph of Theorem 2,*
(b) *$G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$.*

Then there exists a fixed-point-free embedding of G.

Somewhat surprisingly, with only one extra exceptional graph, we have a considerably stronger result (cf. [33]).

Theorem 18. *Let G be a graph of order n with $|E(G)| \leqslant n-1$ and such that*

(a) *G is not an exceptional graph of Theorem 2,*
(b) *$G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$,*
(c) *$G \neq K_1 \cup C_5$.*

Then there exists a cyclic embedding of G.

The cyclic embeddings of (n,n)-graphs are studied in [13].

5. Some labellings of a tree

The main motivation of the results discussed in this section is the following well-known conjecture of Bollobás and Eldridge (see [2]).

Conjecture 19. Let $G_1,\dots,G_k$ be k graphs of order n. If $|E(G_i)| \leqslant n-k$, $i=1,\dots,k$, then $G_1,\dots,G_k$ are packable into K_n.

The case $k=2$ (which was the origin of the conjecture) was proved by Sauer and Spencer in 1978 in [23]. The case $k=3$ was proved recently in [18].

We have already mentioned some results that are related to the special cases of the above conjecture, namely the cases where instead of two or three graphs we can consider two or three copies of the same graph. The aim of this section paper is to consider another special case of the Bollobás and Eldridge conjecture. First of all we set $k=\lfloor n/2 \rfloor$. Observe that in this case the total number of edges we pack into K_n is maximum. Next, because of the methods we use, we consider the case of the packing of k copies of a tree.

A packing of k copies of a graph G will be called a *cyclic packing* of G if there exists a permutation σ on $V(G)$ such that the graphs G, $\sigma(G)$, $\sigma^2(G),\dots,\sigma^{k-1}(G)$ are pairwise disjoint. The main result of the paper [4] can be formulated as follows.

Theorem 20. *Let T be a tree of size $\lceil n/2 \rceil$. Then there exists a cyclic packing of $\lfloor n/2 \rfloor$ copies of T into K_n.*

The main tool in the proof of the above theorem is a special labelling (called: *distinct length labelling* (DLL)) of a graph T. This kind of labelling is defined in the paper; however, graph labellings are well-known and used in decomposition problems such as, for instance, the conjecture of Ringel that the complete graph K_{2k+1} can be decomposed into $2k+1$ subgraphs that are all isomorphic to a given tree with k edges (see e.g. [10]). We introduce some additional terminology. Let K_k be a complete graph with vertex set $\{x_1, x_2, \ldots, x_k\}$. Let G be a graph of order not greater than k. *A DLL* of a graph G in K_k is an injection f from the vertices of G to the set $\{1, 2, \ldots, k\}$ such that, when each edge uv is assigned the label $\min\{f(u) - f(v), f(v) - f(u);\ (\mathrm{mod}\, k)\}$ (with the understanding that we choose residues in $\{0, 1, \ldots, k-1\}$), the resulting edge labels (called: *lengths*) are distinct. Moreover, if k is even we assume that the label $k/2$ does not occur. Thus, there are exactly $\lfloor (k-1)/2 \rfloor$ *admissible* lengths. If we draw G in such a way that the vertex labelled i is identified with x_i, then the label of an edge is the distance between its ends on the cycle $\{x_1, x_2, \ldots, x_k\}$. We shall assume that such an identification has been made. Let $\sigma = (x_1 x_2 \ldots x_k)$ be a cyclic permutation. It is easy to see that the image of an edge e has the same length as e. So, if G has a DLL in K_k, then the permutation σ defines a cyclic packing of k copies of G into K_k.

Remark. Observe that a DLL in K_{2k+1} of a tree of size k would imply the Ringel conjecture. A tree of size k with a DLL using only $k+1$ labels $\{1, 2, \ldots, k+1\}$ is said to be *graceful*. The well-known Ringel–Kotzig Conjecture (Graceful Tree Conjecture) says that all trees are graceful (see [10]).

Let now $K_{k,k}$ be a complete bipartite graph with vertex set partition $L = \{x_1, x_2, \ldots, x_k\}$ and $R = \{y_1, y_2, \ldots, y_k\}$. Let $e = x_i y_j$ be an edge of $K_{k,k}$. The *length* of e is given by $j - i$ modulo k. Let G be a bipartite graph of size not greater than k. *A DLL* of G in $K_{k,k}$ is an injection f from the vertices of G to the set $\{L, R\} \times \{1, 2, \ldots, k\}$ such that

1. for each edge uv the first elements assigned to u and v are distinct, i.e. uv can be considered as an edge of $K_{k,k}$;
2. the lengths of all edges are distinct.

Let $\sigma = (x_1 x_2 \ldots x_k)(y_1 y_2 \ldots y_k)$ be a permutation on vertex set of $K_{k,k}$ having two cycles. It is easy to see that the image of an edge e has the same length as e. So, if G has a DLL in $K_{k,k}$, then the permutation σ defines a cyclic packing of k copies of G into $K_{k,k}$. Observe that in this case there are exactly k *admissible* lengths.

Remark. A DLL in $K_{k,k}$ of a tree of size k has been considered by Ringel et al. [21] as *bigraceful labelling*. They conjecture that all trees have bigraceful labellings which implies that $K_{k,k}$ is decomposable into k copies of any given tree with k edges.

Finally, we shall define yet another labelling. Let now K_{2k} be a complete graph on $2k$ vertices. We partition the vertex set of K_{2k} into two parts $L = \{x_1, x_2, \ldots, x_k\}$ and $R = \{y_1, y_2, \ldots, y_k\}$. *A distinct length labelling* of G into $K_k * K_k$ (the join of two copies of K_k) is an injection f from the vertices of G to the set $\{L,R\} \times \{1,2,\ldots,k\}$ such that

1. f can be considered (in a canonical way) as a DLL in K_k for the subgraph of G induced by the edges having both ends labelled by the pairs with the same first element;
2. f can be considered as a DLL in $K_{k,k}$ for the subgraph of G induced by the edges having the ends labelled by the pairs with distinct first elements.

We shall consider G as a subgraph of K_{2k} and identify the vertices labelled by (L,i) with x_i and the vertices labelled by (R,i) with y_i. This will allow us to use 'geometric' terminology such as, for instance, *crossing edge*. As above, it is easy to see that the permutation $\sigma = (x_1x_2\ldots x_k)(y_1y_2\ldots y_k)$ define a cyclic packing of k copies of G into K_{2k}.

6. Small permutations

Recall that for a graph G and an integer $k \geqslant 2$, the graph G^k is obtained from G by adding edges between all vertices $x, y \in V(G)$ such that $\mathrm{dist}_G(x,y) \leqslant k$.

The following theorem is contained as a lemma in [28] (cf. also [3]).

Theorem 21. *Let T be a non-star tree of order n with $n > 3$. Then there exists an embedding σ of T such that for every $x \in V(T)$, $\mathrm{dist}_T(x,\sigma(x)) \leqslant 3$.*

This theorem immediately implies the following.

Corollary 22. *Let T be a non-star tree of order n with $n > 3$. Then there exists an embedding σ of T such that $\sigma(T) \subset T^7$.*

Since T^7 is, in general, a proper subgraph of K_n, the last corollary can be considered as an improvement of the well-known fact that two copies of a non-star tree of order n can be packed into K_n. Observe that the permutation property that we use is related to the graph that we pack.

Actually, a much stronger result holds (see [19]). We have

Theorem 23. *Let T be a non-star tree of order n with $n > 3$. Then there exists an embedding σ of T such that $\sigma(T) \subset T^4$.*

We shall need some additional definitions and notations in order to define the properties of the permutation used in the proof of this result. Let x be a vertex of a tree T. The components of $T - x$ are called *neighbour trees* of x. If y is any neighbour of x in T, we denote by T_y the neighbour tree of x which contains y. Consequently, if we delete an edge $e = xy$ of T, we obtain two components of $T - e$, respectively the neighbour tree T_x of y and the neighbour tree T_y of x.

Let T be a tree and let x be a vertex of T. A permutation σ on $V(T)$ is said to be (T,x)-*good* iff the following four conditions are satisfied:

1. σ is a 2-placement of T,
2. $\sigma(T) \subset T^4$,
3. $\text{dist}_T(x, \sigma(x)) = 1$,
4. for every neighbour y of the vertex x, $\text{dist}_T(y, \sigma(y)) \leqslant 2$.

The tree itself is said to be x-*good* if there exists a (T,x)-good permutation. Finally, a non-star tree T is called *good* if it is x-good for every x in $V(T)$.

Using this terminology, it is easy to see that Theorem 23 is implied by the following.

Theorem 24. *All non-star trees are good.*

The main idea of the proof is given by the fact that if there exists an edge xy of T such that the neighbour trees T_x and T_y are two good trees, then there exists a (T,x)-good permutation.

Observe that the square of the path of order n contains only $(n-2)$ more edges than the path. So it is not possible, in general, to embed two copies of a tree T into T^2.

On the other hand, we do not know any example of a non-star tree T, two copies of which are not embeddable into T^3. However, in every embedding of the path P_7 in its third power, the centre of the path remains fixed. So, in order to prove that, for every non-star tree T, there exists an embedding of two copies of T into T^3, we need other technics that those used above.

Some special families of trees and their packing in their third powers have been considered recently by Kheddouci [17].

References

[1] A. Benhocine, A.P. Wojda, On self-complementation, J. Graph Theory 8 (1985) 335–341.
[2] B. Bollobás, Extremal Graph Theory, Academic Press, London, 1978.
[3] S. Brandt, Embedding graphs without short cycles in their complements, in: Y. Alavi, A. Schwenk (Eds.) Graph Theory, Combinatorics, and Applications of Graphs, Vol. 1, Wiley, New York, 1995, pp. 115–121.
[4] S. Brandt, M. Woźniak, On cyclic packing of a tree, submitted for publication.
[5] D. Burns, S. Schuster, Every $(p, p-2)$ graph is contained in its complement, J. Graph Theory 1 (1977) 277–279.
[6] D. Burns, S. Schuster, Embedding $(n, n-1)$ graphs in their complements, Israel J. Math. 30 (1978) 313–320.
[7] Y. Caro, R. Yuster, Packing graphs: the packing problem solved, Electron. J. Combin. 4 (1) (1997) (Research Paper 1).
[8] C.R.J. Clapham, Graphs self-complementary in $K_n - e$, Discrete Math. 81 (1990) 229–235.
[9] R.J. Faudree, C.C. Rousseau, R.H. Schelp, S. Schuster, Embedding graphs in their complements, Czechoslovak Math. J. 31 (106) (1981) 53–62.
[10] J.A. Gallian, A dynamic survey on graph labelling, www.combinatorics.org/.
[11] T. Gangopadhyay, Packing graphs in their complements, Discrete Math. 186 (1998) 117–124.
[12] R.A. Gibbs, Self-complementary graphs, J. Combin. Theory (B) 16 (1974) 106–123.

[13] A. Görlich, M. Pilśniak, M. Woźniak, On cyclically embeddable (n,n)-graphs, Discuss. Math. Graph Theory 23 (2003) 85–104.
[14] F. Harary, R.W. Robinson, N.C. Wormald, Isomorphic factorisations. I: complete graphs, Trans. Amer. Math. Soc. 242 (1978) 243–260.
[15] T. Hasunuma, Y. Shibata, Remarks on the placeability of isomorphic trees in a complete graph, J. Graph Theory 21 (1) (1996) 41–42.
[16] C. Huang, A. Rosa, Decomposition of complete graphs into trees, Ars Combin. 5 (1978) 26–63.
[17] H. Kheddouci, Packing some trees into their third power, Rapport de Recherche LRI, 1998.
[18] H. Kheddouci, S. Marshall, J-F. Saclé, M. Woźniak, On the packing of three graphs, Discrete Math. 236 (2001) 197–225.
[19] H. Kheddouci, J-F. Saclé, M. Woźniak, Packing two copies of a tree into its fourth power, Discrete Math. 213 (2000) 169–178.
[20] G. Ringel, Selbstkomplementäre Graphen, Arch. Math. 14 (1963) 354–358.
[21] G. Ringel, A. Llado, O. Serra, Decomposition of complete bipartite graphs into trees, DMAT Research Report, Vol. 11, Univ. Politecnica de Catalunya, 1996.
[22] H. Sachs, Über selbstkomplementäre Graphen, Publ. Math. Debrecen 9 (1970) 270–288.
[23] N. Sauer, J. Spencer, Edge disjoint placement of graphs, J. Combin. Theory Ser. B 25 (1978) 295–302.
[24] S. Schuster, Fixed-point-free embeddings of graphs in their complements, Internat. J. Math. Math. Sci. 1 (1978) 335–338.
[25] Z. Skupień, The complete graph t-packings and t-coverings, Graphs Combin. 9 (1993) 353–363.
[26] H. Wang, N. Sauer, Packing three copies of a tree into a complete graph, European J. Combin. 14 (1993) 137–142.
[27] A.P. Wojda, M. Woźniak, I.A. Zioło, On self-complementary supergraphs of (n,n)-graphs, submitted for publication.
[28] M. Woźniak, A note on embedding graphs without small cycles, Colloq. Math. Soc. J. Bolyai 60 (1991) 727–732.
[29] M. Woźniak, Embedding graphs of small size, Discrete Appl. Math. 51 (1994) 233–241.
[30] M. Woźniak, Packing three trees, Discrete Math. 150 (1996) 393–402.
[31] M. Woźniak, Packing of Graphs, Dissertationes Math. 362 (1997) 1–78.
[32] M. Woźniak, On cyclically embeddable graphs, Discuss. Math. Graph Theory 19 (2) (1999) 241–248.
[33] M. Woźniak, On cyclically embeddable $(n,n-1)$ graphs, Discrete Math. 251 (2002) 173–179.
[34] M. Woźniak, A.P. Wojda, Triple placement of graphs, Graphs Combin. 9 (1993) 85–91.
[35] H.P. Yap, Some Topics In Graph Theory, in: London Mathematical Society, Lectures Notes Series, Vol. 108, Cambridge University Press, Cambridge, 1986.
[36] H.P. Yap, Packing of graphs—a survey, Discrete Math. 72 (1988) 395–404.