Packing of graphs
— some recent results and trends

Mariusz Woźniak *
AGH University of Science and Technology
Faculty of Applied Mathematics
Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: mwozniak@agh.edu.pl

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Abstract

We discuss some results, some quite recent, concerning the packing of graphs. We are interested mainly in the packing into a planar graph as well as in packing in transitive tournaments.

1 Introduction

We shall use standard graph theory notation. Let G be a finite, undirected graph $G = (V(G), E(G))$ of order $n = |V(G)| =: |G|$ and size $|E(G)| =: \|G\|$. All graphs will be assumed to have neither loops nor multiple edges. If a graph G has order n and size m, we say that G is an (n, m)-graph.

Suppose $G_1, \cdots, G_k$ are graphs of order less or equal to n. We say that there is a *packing* of $G_1, \cdots, G_k$ (into the complete graph K_n) if there exist injections $\alpha_i : V(G) \longrightarrow V(K_n),\ i = 1, \ldots, k$, such that

$$\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G)) \neq \emptyset \text{ for } i \neq j,$$

where the map $\alpha_i^* : E(G_i) \longrightarrow E(K_n)$ is induced by α_i.

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A packing of k copies of a graph G will be called a *k-placement* of G. A packing of two copies of G, i.e. a 2-placement, is an *embedding* of G (in its complement $\overline{G}$). So, an embedding of a graph G is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

It is easy to see that there exists a packing of two graphs G_1 and G_2 if and only if G_1 is a subgraph of $\overline{G_2}$ (or, by symmetry, G_2 is a subgraph of $\overline{G_1}$). However, we wish to distinguish between the problems about packings and the problems about the existence of certain subgraphs. In 'packing' problems we claim that each member of a **large** family of graphs contains each member of another **large** family. In 'subgraph' problems usually at least one of two graphs is fixed.

The following theorem was proved, independently, in [2], [3] and [14].

Theorem 1 *Let $G = (V, E)$ be a graph of order n. If $\|G\| \le n-2$ then G can be embedded in its complement.* ∎

Theorem 1 can be considered as a special case of the following theorem.

Theorem 2 *Let G and H be two graphs of order n. If $\|G\| \le n-2$ and $\|H\| \le n-2$ then G and H can be packed into K_n.* ∎

The example of the star S_n shows that neither Theorem 1 nor Theorem 2 can be improved by raising the size of G even in the case when G is a tree. However in this case we have the following two theorems. The second of these theorems is a generalisation of the first one, however it will be convenient for us to formulate them separately.

Theorem 3 *Let T be a non-star tree of order n. Then T can be embedded into its complement.* ∎

Theorem 4 *Let T_1 and T_2 be two non-star trees of order n. Then T_1 and T_2 can be packed into K_n.* ∎

These results has been improved in many ways. The main references of the paper and of other packing problems are the last chapter of Bollobás's book [2], the 4th Chapter of Yap's book [21] and the survey papers [22] and [18]. The recent author's survey [19] concern these improvements of

Theorem 1 that deal with some additional properties of packing permutations such that, for instance, the cyclic structure of the packing permutation.

In this short article we are interested mainly with two relatively new directions in the study of packing of graphs. In Section 2 we discuss so called *planar* packing, and in Section **??** we consider the packing of orgraphs into transitive tournaments.

We shall need some additional definitions.

Let G_1 and G_2 be graphs with $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. The *edge sum* $G_1 \oplus G_2$ has $V(G) = V(G_1) = V(G_2)$ as the set of vertices and $E(G) = E(G_1) \cup E(G_2)$ as the set of edges.

A graph G is *self-complementary* (briefly, s-c) if it is isomorphic to its complement (cf. [11], [12], or [5]). It is clear that an s-c graph has $n \equiv 0, 1 \pmod 4$ vertices. (We can extend the above definition also to the case where $n \equiv 2, 3 \pmod 4$. We speak then about *almost self-complementary* graphs).

We are interested in self-complementary graphs because it is evident that subgraphs of self-complementary graphs are embeddable. It is known that for (n, k)-graphs with $k \leq n$ the property 'to be embeddable' and the property 'to be a subgraph of a self-complementary graph' are in fact equivalent (see e.g. [1], [17] or [16]).

2 Packing two copies of a tree into a planar graph

One on the most interesting new directions in packing theory is the following problem: can the packing of two packable graphs be chosen in such a way that the resulting graph is planar? This problem was first considered by A. Garcia, C. Hernando, F. Hurtado, M. Noy and J. Tejel [4] in the case of two trees.

The fact that two copies of a non-star tree T of order n can be packed into the complete graph K_n is well known. However, it may happen that the graph $T \oplus \sigma(T)$ is not planar. For instance, in the packing given in Fig. 2 it is easy to see that the graph $T \oplus \sigma(T)$ contains a subdivision of K_5. Therfore, it is not planar. The packing of the same tree into a planar graph in given Fig. 2.

The above mentioned authors proved the following theorem.

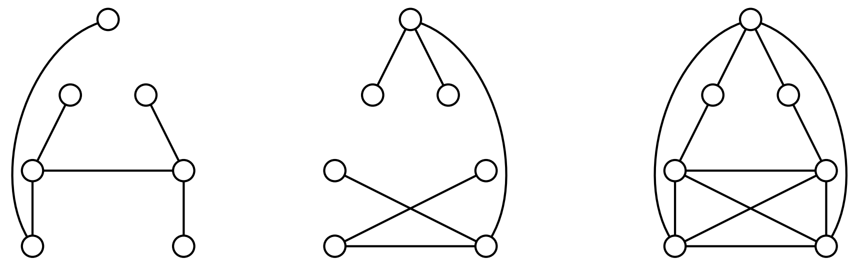

Figure 1: A tree T, its copy and a packing of them with a non-planar $T \oplus \sigma(T)$.

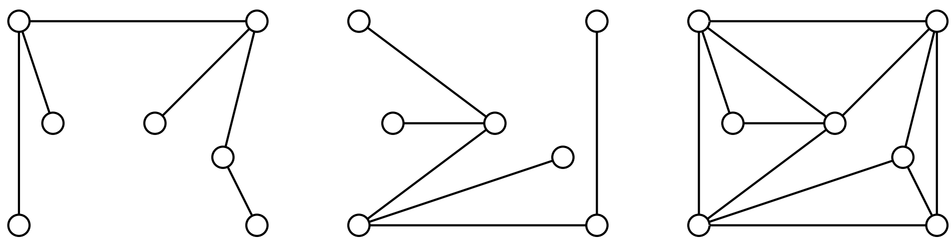

Figure 2: A tree T, its copy and a packing of them into a planar graph $T \oplus \sigma(T)$.

Theorem 5 *If T is a non-star tree, then there exists a 2-pacting of T such that the graph induced by the edges of these two copies is planar.*

Another, somewhat shorter than the original one, proof of the above theorem can be found in [20]. This proof is based on the cyclic packing of a non-star tree.

The authors of [4] conjectured that the above property holds also in the case when we pack two arbitrary non-star trees.

3 Packing problems in transitive tournaments

Let $\overrightarrow{G}$ be an oriented graph (*orgraph*) of order n with vertex set $V(\overrightarrow{G})$ and arc set $E(\overrightarrow{G})$. Then $\overrightarrow{G}$ can be viewed either as a simple graph with a given orientation on the edges or as a directed graph in which between every two distinct vertices in $V(\overrightarrow{G})$ at most one arc occurs.

Let $\overrightarrow{G}(V, E)$ be an oriented graph. An *underlying graph* G is an undi-

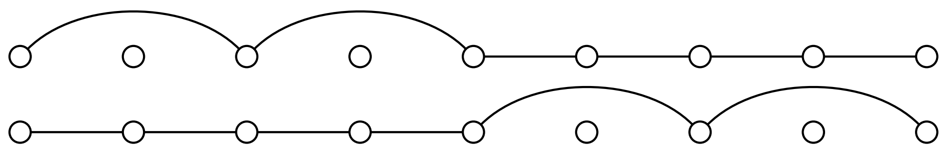

Figure 3: Two packing of an oriented path of length six into TT_9. All edges (arcs) are oriented from the left to the right.

rected graph on the same set of vertices $V(G)$ and $e \in E(G)$ if and only if $\vec{e} \in E(\overrightarrow{G})$.

A *tournament* is a complete oriented graph, that is a directed graph in which for any distinct vertices x, y there is an arc from x to y or there is an arc from y to x.

A *transitive tournament* on n vertices, denoted by TT_n, is a transitively oriented clique of size n i.e. if uv and vw are arcs in TT_n, then uw is also arc in TT_n.

An *(oriented) path* between two distinct vertices u and v in an oriented graph $\overrightarrow{G}$ is a finite sequence $u = v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k = v$ of vertices and arcs, beginning with u and ending with v such that $e_i = v_{i-1}v_i \in E(!\overrightarrow{G})$ for $i = 1, \ldots, k$. The number k is the length of the path. A *semi-path* between two distinct vertices u and v is a path between u and v in an underlying graph G. Analogously, we define an (oriented) cycle and semi-cycle.

There is a well known fact that each tournament contains a Hamiltonian path. In TT_n this path is unique. It implies that we can identify the vertex set of TT_n with the set $|1, 2, \ldots, n|$ in such a way that the arc $ij \in E(TT_n)$ if and only if $i < j$.

An arc (i, j) is said of length $|i - j| = m$.

Let $\overrightarrow{G}$ be an oriented graph such that $|V(\overrightarrow{G})| = n$. An *embedding of* $\overrightarrow{G}$ *in* TT_n is a pair of arc disjoint subgraphs of TT_n, $\{\overrightarrow{H_1}, \overrightarrow{H_2}\}$, such that $\overrightarrow{H_1} \cong \overrightarrow{G}$, $\overrightarrow{H_2} \cong \overrightarrow{G}$. It means that $\overrightarrow{G}$ is a subgraph of $\overrightarrow{H} := TT_n \setminus \overrightarrow{G}$, its complement in TT_n or we can say that there exist a permutation $\sigma : V(\overrightarrow{G}) \to V(\overrightarrow{G})$ such that if an arc uv belongs to $E(\overrightarrow{G})$ then the arc $\sigma(u)\sigma(v)$ does not belong to $E(\overrightarrow{G})$.

In 1999 Sali and Simonyi proved in [13] that any self-complementary graph

H on order n can be oriented in such a way, that the graph $H \oplus \sigma(H)$ is isomorphic to the graph TT_n (σ denotes the self-complementary permutation). A short proof of this fact was given by Gyárfás [8]. Since there are many results concerning the relationship between packing and self-complementary graphs, the above mentioned result suggests that one could get some non-trivial results by studying the packing problems in transitive tournaments.

As observed by A.P.Wojda ([15]) there are tournaments which do not seem to be very interesting from the point of view of packing theory. For instance, let us consider the tournament obtained from a TT_n by changing the orientation of the arc $1n$. This tournament contains $n-2$ oriented triangles but do not contain two arc-disjoint triangles.

Probably, the first result dealing with packing in transitive tournaments is the following theorem:

Theorem 6 *Let TT_n be a transitive tournament on n vertices. Let $\overrightarrow{G}$ be a subgraph of TT_n such that $|E(\overrightarrow{G})| \leq \frac{3(n-1)}{4}$. Then $\overrightarrow{G}$ is embeddable in TT_n.* ■

The proof of the above theorem is given in [7] Here, we shall show only that the condition on the size of the graph cannot be weakened. This will allow us to present how the specific properties of TT_n (with respect to the complete graph) can be used in packing theory problems.

Claim. The condition on the size of the graph in Theorem 6 cannot be weakened.

Proof. Let us consider a path of length k. Let us suppose that there exists an embedding of such path in TT_n, where $n > k$. It means that there exist $\overrightarrow{G}$ and $\overrightarrow{G}'$, two arc disjoint subgraphs of transitive tournament TT_n isomorphic to such path. Let k_1 and k_1' denote the number of arcs in $\overrightarrow{G}$ and $\overrightarrow{G'}$ of length one and k_2 and k_2' denote the number of arcs of length greater than one, respectively. Thus $k_1 + k_2 = k$ and $k_1' + k_2' = k$.

Since $\overrightarrow{G}$ and $\overrightarrow{G'}$ are the subgraphs of TT_n we have $k_1 + 2k_2 \leq n-1$ and $k_1' + 2k_2 \leq n-1$.

By adding two above inequalities we get $k_1 + k_1' + 2k_2 + 2k_2' \leq 2n-2$.

On the other hand, since $\overrightarrow{G}$ and $\overrightarrow{G'}$ are arc disjoint and the total number of arc of length one in TT_n is equal to $(n-1)$ we have $k_1 + k_1' \leq n-1$.

By adding two last inequalities and using above equalities we get $4k \leq 3n - 3$, which finishes the proof. ■

Recently, some other results were obtained by M.Pilśniak. We cite here two of them.

Theorem 7 *[9] Let $\overrightarrow{G}$ be a subgraph of TT_n (a transitive tournament on n vertices) such that $|E(\overrightarrow{G})| \leq \frac{2}{3}n - 1$. Then the is a packing of three copies of $\overrightarrow{G}$ into TT_n.*

Theorem 8 *[10] Suppose that $\overrightarrow{G}$ and $\overrightarrow{H}$ are two subgraphs of TT_n (a transitive tournament on n vertices). If $|E(\overrightarrow{G})| + |E(\overrightarrow{H})| \leq \frac{3(n-1)}{2}$, then $\overrightarrow{G}$ and $\overrightarrow{G}$ are packable into TT_n.*

Finally, mention by the way that also some decomposition problems in TT_n have been considered (see [6]).

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